

CHAPTER 1

Preliminaries

§1 Set Theory

We assume that the reader is familiar with basic set theory. In this paragraph, we want to recall the relevant definitions and fix the notation.

Our approach to set theory will be informal. For our purposes, a *set* is a collection of objects, taken as a whole. "Set" is therefore a collective term like "family", "flock", "species", "army", "club", "team" etc. The objects which make up a set are called the *elements* of that set. We write

$$x \in S$$

to denote that the object x is an element of the set S . This can be read " x is an element of S ", or " x is a member of S ", or " x belongs to S ", or " x is in S ", or " x is contained in S ", or " S contains x ". If x is not an element of S , we write

$$x \notin S.$$

For technical reasons, we agree to have a unique set that has no elements at all. This set is called the *empty set* and is denoted by \emptyset .

A set S is called a *subset* of a set T if every element of S is also an element of T . The notation

$$S \subseteq T$$

means that S is a subset of T . This is read " S is a subset of T ", or " S is included in T ", or " S is contained in T ". By convention, the empty set \emptyset is a subset of any set. If S is not a subset of T , we write

$$S \not\subseteq T.$$

This means there is at least one element of S which does not belong to T .

If $S \subseteq T$ and $T \subseteq S$, then S and T have exactly the same elements. In this case, S and T are said to be *identical* or *equal*. We write

$$S = T$$

if S and T are equal sets. Whenever we want to prove that two sets S and T are equal, we must show that S is included in T and that T is included in S . If S and T are not equal, we put

$$S \neq T.$$

If $S \subseteq T$ but $T \neq S$, then S is said to be a *proper subset* of T . So S is a proper subset of T if and only if every element of S is an element of T but T contains at least one element which does not belong to S . The notation

$$S \subset T$$

means that S is a proper subset of T . This is read " S is a proper subset of T ", or " S is properly included in T ", or " S is properly contained in T ". By convention, the empty set \emptyset is a proper subset of every set except itself.

Some authors write $S \subset T$ to mean that S is a subset of T , the possibility $S = T$ being included, and $S \subsetneq T$ to mean that S is a proper subset of T . The reader should be careful about the meaning of the symbol " \subset " he or she uses. In this book, " \subset " denotes proper inclusion.

Sets are sometimes written by displaying their elements within braces (roster notation). Hence

$$\{1,2,3,4,5\}$$

is the set whose elements are the numbers 1,2,3,4 and 5. Obviously, only those sets which have a small number of elements can be written in this

way. In many cases, the elements of a set S are characterized by a property P and the set is then written

$$\{x : x \text{ has property } P\}.$$

In this book, $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers, $\mathbb{Z} = \{0, \mp 1, \mp 2, \dots\}$ is the set of integers, $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ is the set of rational numbers, \mathbb{R} is the set of real numbers, \mathbb{C} is the set of complex numbers. These notations are standard. Some authors regard 0 as a natural number, but we agree that $0 \notin \mathbb{N}$ in this book.

Given two sets S and T , we consider those objects which belong to S or to T . Such objects will make up a new set. This set is called the *union of S and T* and is denoted by $S \cup T$. We remark here that 'or' in the definition of a union is the logical 'or'. Let us recall that

$$\begin{aligned} 'p \text{ or } q' \text{ is true in case } & 'p' \text{ is true, } 'q' \text{ is true;} \\ & 'p' \text{ is true, } 'q' \text{ is false;} \\ & 'p' \text{ is false, } 'q' \text{ is true;} \end{aligned}$$

and

$$'p \text{ or } q' \text{ is false in case } 'p' \text{ is false, } 'q' \text{ is false.}$$

Thus we have

$$S \cup T = \{x : x \in S \text{ or } x \in T\}.$$

In particular, $S \cup T = T \cup S$.

If we have sets S_1, S_2, \dots, S_n , their union $S_1 \cup S_2 \cup \dots \cup S_n$ is given by

$$S_1 \cup S_2 \cup \dots \cup S_n = \{x : x \in S_1 \text{ or } x \in S_2 \text{ or } \dots \text{ or } x \in S_n\}.$$

We usually contract this notation into $\bigcup_{i=1}^n S_i$, just like we write $\sum_{i=1}^n a_i$

instead of $a_1 + a_2 + \dots + a_n$. More generally, if we have sets S_i , indexed by a set I , then their union $\bigcup_{i \in I} S_i$ is the set

$$\bigcup_{i \in I} S_i = \{x : x \in S_i \text{ for at least one } i \in I\}.$$

Given two sets S and T , we consider those objects which belong to S and to T . Such objects will make up a new set. This set is called the *intersection of S and T* and is denoted by $S \cap T$. We remark here that 'and' in the definition of a intersection is the logical 'and'. Let us recall that

$$'p \text{ and } q' \text{ is true in case } 'p' \text{ is true, } 'q' \text{ is true;}$$

and

' p and q ' is false in case ' p ' is true, ' q ' is false;
' p ' is false, ' q ' is true;
' p ' is false, ' q ' is false.

Thus we have

$$S \cap T = \{x : x \in S \text{ and } x \in T\}.$$

In particular, $S \cap T = T \cap S$.

If we have sets S_1, S_2, \dots, S_n , their intersection $S_1 \cap S_2 \cap \dots \cap S_n$ is given by

$$S_1 \cap S_2 \cap \dots \cap S_n = \{x : x \in S_1 \text{ and } x \in S_2 \text{ and } \dots \text{ and } x \in S_n\}.$$

We usually contract this notation into $\bigcap_{i=1}^n S_i$. More generally, if we have sets S_i , indexed by a set I , then their intersection $\bigcap_{i \in I} S_i$ is the set

$$\bigcap_{i \in I} S_i = \{x : x \in S_i \text{ for all } i \in I\}.$$

Two sets S and T are said to be *disjoint* if their intersection is empty: $S \cap T = \emptyset$. Given a family of sets S_i , indexed by a set I , the sets S_i are called *mutually disjoint* if any two distinct of them are disjoint:

$$S_{i_1} \cap S_{i_2} = \emptyset \text{ for all } i_1, i_2 \in I, S_{i_1} \neq S_{i_2}.$$

The sets we consider in a particular discussion are usually subsets of a set U . This set U is called the *universal set*. Given a set S , which is a subset of a universal set U , those elements of U that do not belong to S make up a new set, called the *complement* of S and denoted by S^c or S^{\complement} or $C_U(S)$. Hence

$$S^c = \{x : x \in U \text{ and } x \notin S\}.$$

More generally, we write

$$T \setminus S = \{x : x \in T \text{ and } x \notin S\}.$$

and call this set the *relative complement of S in T* , or the *difference set T minus S* . The set S may or may not be a subset of T . Note that

$$T \setminus S = T \cap S^c.$$

According to our definition of equality, the sets $\{a, b\}$ and $\{b, a\}$ are equal. Frequently, we want to distinguish between a, b and b, a . To this end, we define ordered pairs. An *ordered pair* is a pair of objects a, b , enclosed

within parentheses and separated by a comma. Thus (a,b) is an ordered pair. The adjective "ordered" is used to emphasize that the objects have a status of being first and being second. a is called the *first component* of the ordered pair (a,b) , and b is called its *second component*. Two ordered pairs are declared equal if their first components are equal and their second components are equal. Thus (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$, in which case we write $(a,b) = (c,d)$. Notice that we have $(a,b) \neq (b,a)$ unless $a = b$ (here \neq means the negation of equality).

The set of all ordered pairs, whose first components are the elements of a set S and whose second components are the elements of a set T , is called the *cartesian product of S and T* , and is denoted by $S \times T$. Hence

$$S \times T = \{(a,b): a \in S \text{ and } b \in T\}.$$

We can also define ordered triples (a,b,c) , ordered quadruples (a,b,c,d) , more generally ordered n -tuples (a_1, a_2, \dots, a_n) . Equality of ordered n -tuples will mean the equality of their corresponding components. The set of all ordered n -tuples, whose i -th components are the elements of a set S_i , is called the *cartesian product of S_1, S_2, \dots, S_n* and is denoted by $S_1 \times S_2 \times \dots \times S_n$. Hence

$$S_1 \times S_2 \times \dots \times S_n = \{(a_1, a_2, \dots, a_n): a_1 \in S_1, a_2 \in S_2, \dots, a_n \in S_n\}.$$

It is possible to define the cartesian product of infinitely many sets, too. We do not give this definition, for we will not need it.

A set can have finitely many or infinitely many elements. The number of elements in a set S is called the *cardinality* or the *cardinal number of S* . The cardinality of S is denoted by $|S|$. The set S is said to be *finite* if $|S|$ is a finite number. S is said to be *infinite* if S is not finite. A rigorous definition of finite and infinite sets must be based on the notion of one-to-one correspondence between sets, which will be introduced in §3. However, we will not make any attempt to give a rigorous definition of finite and infinite sets. We shall be content with the suggestive description above.

Exercises

1. Show that, if R is a subset of S and S is a subset of T , then R is a subset of T .

2. Show that $(R \cup S) \cup T = R \cup (S \cup T)$
 and $(R \cap S) \cap T = R \cap (S \cap T)$.
3. Prove: $S \cap T = S$ if and only if $S \subseteq T$, and $S \subseteq T$ if and only if $S \cup T = T$.
4. Prove the distributivity of union over intersection and of intersection over union:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T),$$

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

5. Prove the deMorgan laws:

$$(S \cup T)^c = S^c \cap T^c \quad \text{and} \quad (S \cap T)^c = S^c \cup T^c$$

for any subsets S, T of a universal set U .

6. Show that $T \setminus \emptyset = T$ and $T \setminus T = \emptyset$ for any set T .
7. Prove: $(S \setminus T) \cup (T \setminus S) = (S \cup T) \setminus (S \cap T)$. This set is called the *symmetric difference of S and T* . It is denoted by $S \Delta T$.
8. With the notation of Ex. 7, prove that

$$(R \Delta S) \Delta T = R \Delta (S \Delta T)$$

$$S \Delta \emptyset = S$$

$$S \Delta S = \emptyset$$

$$S \Delta T = T \Delta S.$$

9. Let S and T be finite sets. Prove the following assertions.
- a) If $S \cap T = \emptyset$, then $|S \cup T| = |S| + |T|$.
- b) $|S \cup T| = |S| + |T| - |S \cap T|$. (Hint: $S \cup T = S \cup (T \setminus S)$.)
10. Find all subsets of $\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}$.
11. Prove: if S is a finite set, then S has exactly $2^{|S|}$ subsets.