

§2 Equivalence Relations

In mathematics, we often investigate relationships between certain objects (numbers, functions, sets, figures, etc.). If an element a of a set A is related to an element b of a set B , we might write

a is related to b

or shortly

a related b

or even more shortly

$a R b$.

The essential point is that we have two objects, a and b , that are related in some way. Also, we say " a is related to b ", not " b is related to a ", so the *order* of a and b is important. In other words, the ordered pair (a,b) is distinguished by the relation. This observation suggests the following formal definition of a relation.

2.1 Definition: Let A and B be two sets. A *relation R from A into B* is a subset of the cartesian product $A \times B$.

If A and B happen to be equal, we speak of a relation *on A* instead of using the longer phrase "a relation from A into A ".

Equivalence relations constitute a very important type of relations on a set.

2.2 Definition: Let A be a nonempty set. A relation R on A (that is, a subset R of $A \times A$) is called an *equivalence relation on A* if the following hold.

- (i) $(a,a) \in R$ for all $a \in A$,
- (ii) if $(a,b) \in R$, then $(b,a) \in R$ (for all $a,b \in A$)
- (iii) if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

(for all $a,b,c \in R$).

This definition presents the logical structure of an equivalence relation very clearly, but we will almost never use this notation. We prefer to write $a \sim b$, or $a \approx b$, or $a \equiv b$ or some similar symbolism instead of $(a,b) \in R$ in order to express that a,b are related by an equivalence relation R . Here $a \sim b$ can be read " a is equivalent to b ". Our definition then assumes the form below.

2.2 Definition: Let A be a nonempty set. A relation R on A (that is, a subset R of $A \times A$) is called an *equivalence relation on A* if the following hold.

- (i) $a \sim a$ for all $a \in A$,
- (ii) if $a \sim b$, then $b \sim a$ (for all $a,b \in A$),
- (iii) if $a \sim b$ and $b \sim c$ then $a \sim c$ (for all $a,b,c \in A$).

A relation \sim that satisfies the first condition (i) is called a *reflexive* relation, one that satisfies the second condition (ii) is called a *symmetric* relation, one that satisfies the third condition (iii) is called a *transitive* relation. An equivalence relation is therefore a relation which is reflexive, symmetric and transitive. Notice that symmetry and transitivity requirements involve conditional statements (if \dots , then \dots). In order to show that \sim is symmetric, for example, we must make the hypothesis $a \sim b$ and use this hypothesis to establish $b \sim a$. On the other hand, in order to show that \sim is reflexive, we have to establish $a \sim a$ for all $a \in A$, without any further assumption.

2.3 Examples: (a) Let A be a nonempty set of numbers and let equality $=$ be our relation. Then $=$ is certainly an equivalence relation on A since

- (i) $a = a$ for all $a \in A$,
- (ii) if $a = b$, then $b = a$ (for all $a,b \in A$),
- (iii) if $a = b$ and $b = c$ then $a = c$ (for all $a,b,c \in A$).

(b) Let A be the set of all points in the plane except the origin. For any two points P and R in A , let us put $P \sim R$ if R lies on the line through the origin and P .

(i) $P \sim P$ for all points P in A since any point lies on the line through the origin and itself. Thus \sim is reflexive.

(ii) If $P \sim R$, then R lies on the line through the origin and P ; therefore the origin, P , R lie on one and the same line; therefore P lies on the line through the origin and R ; and $R \sim P$. Thus \sim is symmetric.

(iii) If $P \sim R$ and $R \sim T$, then the line through the origin and R contains the points P and T , so T lies on the line through the origin and P , so we get $P \sim T$. Thus \sim is transitive.

This proves that \sim is an equivalence relation on A .

(c) Let S be the set of all straight lines in the plane. Let us put $m//n$ if the line m is parallel to the line n . It is easily seen that $//$ (parallelism) is an equivalence relation on S .

(d) Let \mathbb{Z} be the set of integers. For any two numbers a, b in \mathbb{Z} , let us put $a \equiv b$ if $a - b$ is even (divisible by 2).

(i) $a \equiv a$ for all $a \in \mathbb{Z}$ since $a - a = 0$ is an even number.

(ii) If $a \equiv b$, then $a - b$ is even, then $b - a = -(a - b)$ is also even, so $b \equiv a$.

(iii) If $a \equiv b$ and $b \equiv c$, then $a - b$ and $b - c$ are even. Their sum is also an even number. So $a - c = (a - b) + (b - c)$ is even and $a \equiv c$.

We see that \equiv is an equivalence relation on \mathbb{Z} .

(e) The last example may be generalized. We fix a whole number $n \neq 0$ (n is called the *modulus* in this context). For any two numbers a, b in \mathbb{Z} , let us put $a \equiv b$ if $a - b$ is divisible by n .

(i) $a \equiv a$ for all $a \in \mathbb{Z}$ since $a - a = 0 = n \cdot 0$ is divisible by n .

(ii) If $a \equiv b$, then $a - b = nm$ for some $m \in \mathbb{Z}$, so

$b - a = -(a - b) = n(-m)$ is divisible by n , and $b \equiv a$.

(iii) If $a \equiv b$ and $b \equiv c$, then $a - b = nm$ and $b - c = nk$ for some $m, k \in \mathbb{Z}$, so $a - c = (a - b) + (b - c) = nm + nk = n(m + k)$ is divisible by n , and so $a \equiv c$.

Therefore, \equiv is an equivalence relation on \mathbb{Z} . This relation is called *congruence*. For each nonzero integer n , there is a congruence relation. In order to distinguish between them, we write, when n is the modulus, $a \equiv b \pmod{n}$ rather than $a \equiv b$.

(f) Let $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Thus S is the set of all ordered pairs of integers whose second components are distinct from zero. Let us write

$(a, b) \approx (c, d)$ for $(a, b), (c, d) \in S$ if $ad = bc$.

(i) $(a, b) \approx (a, b)$ for all $(a, b) \in S$, since $ab = ba$ for all $a \in \mathbb{Z}$, $b \in \mathbb{Z} \setminus \{0\}$.

(ii) If $(a,b) \approx (c,d)$, then $ad = bc$, then $da = cb$, then $cb = da$, so $(c,d) \approx (a,b)$.

(iii) If $(a,b) \approx (c,d)$ and $(c,d) \approx (e,f)$, then

$$ad = bc \quad \text{and} \quad cf = de$$

$$df = bcf \quad \text{and} \quad bcf = bde$$

$$adf = bde$$

$$d(af - be) = 0$$

$$af - be = 0 \quad (\text{since } d \neq 0)$$

$$af = be$$

$$(a,b) \approx (e,f).$$

Thus \approx is an equivalence relation on S .

(g) Let T be the set of all triangles in the Euclidean plane. Congruence of triangles is an equivalence relation on T .

(h) Let S be the set of all continuous functions defined on the closed interval $[0,1]$. For any two functions f, g in S , let us write $f \stackrel{\Delta}{=} g$ if

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx.$$

Then $\stackrel{\Delta}{=}$ is an equivalence relation on S .

An equivalence relation is a weak form of equality. Suppose we have various objects, which are similar in one respect and dissimilar in certain other respects. We may wish to ignore their dissimilarity and focus our attention on their similar behaviour. Then there is no need to distinguish between our various objects that behave in the same way. We may regard them as equal or identical. Of course, "equal" or "identical" are poor words to employ here, for the objects are not absolutely identical, they are equal only in one respect that we wish to investigate more closely. So we employ the word "equivalent". That a and b are equivalent means, then, a and b are equal, not in every respect, but rather as far as a particular property is concerned. An equivalence relation is a formal tool for disregarding differences between various objects and treating them as equals.

Let us examine our examples under this light. In Example 2.3(b), the points P and R may be different, but the lines they determine with the origin are equal. In Example 2.3(c), the lines may be different, but their directions are equal. In Example 2.3(d), the integers may be different, but their parities are equal. In Example 2.3(e), the integers may be

different, but their remainders, when they are divided by n , are equal. In Example 2.3(f), the pairs may be different, but the ratio of their components are equal. In Example 2.3(g), the triangles may have different locations in the plane, but their geometrical properties are the same. In Example 2.3(h), the functions may be different, but the "areas under their curves" are equal.

An equivalence relation \sim on a set A gives rise to a partition of A into disjoint subsets. This means that A is a union of certain subsets of A and that the distinct subsets here are mutually disjoint. The converse is also true: whenever we have a partition of a nonempty set A into pairwise disjoint subsets, there is an equivalence relation on A . Before proving this important result, we introduce a definition.

2.4 Definition: Let \sim be an equivalence relation on a nonempty set A , and let a be an element of A . The *equivalence class of a* is defined to be the set of all elements of A that are equivalent to a .

The equivalence class of a will be denoted by $[a]$ (or by $\text{class}(a)$, $\text{cl}(a)$, ba or by a similar symbol): $[a] = \{ x \in A : x \sim a \}$.

An element of an equivalence class $X \subseteq A$ is called a *representative of X* . Notice that $x \in [a]$ and $x \sim a$ have exactly the same meaning. In particular, we have $a \in [a]$ by reflexivity. So any $a \in A$ is a representative of its own equivalence class.

The equivalence classes $[a]$ are subsets of A . The set of all equivalence classes is sometimes denoted by A/\sim . It will be a good exercise for the reader to find the equivalence classes in Example 2.3.

We now state and prove the result we promised.

2.5 Theorem: *Let A be a nonempty set and let \sim be an equivalence relation on A . Then the equivalence classes form a partition of A . In other words, A is the union of the equivalence classes and the distinct equivalence classes are disjoint:*

$$A = \bigcup_{a \in A} [a] \quad \text{and} \quad \text{if } [a] \neq [b], \text{ then } [a] \cap [b] = \emptyset.$$

Conversely, let

$$A = \bigcup_{a \in A} P_i, \quad P_i \cap P_j = \emptyset \quad \text{if } i \neq j$$

be a union of nonempty, mutually disjoint sets P_i indexed by I . Then there is an equivalence relation on A such that the P_i 's are the equivalence classes under this relation.

Proof: First we prove $A = \bigcup_{a \in A} [a]$. For any $a \in A$, we have $[a] \subseteq A$, hence $\bigcup_{a \in A} [a] \subseteq A$. Also, if $a \in A$, then $a \in [a]$ by reflexivity, so $a \in \bigcup_{a \in A} [a]$ and $A \subseteq \bigcup_{a \in A} [a]$. So $A = \bigcup_{a \in A} [a]$.

Now we must prove that distinct equivalence classes are disjoint. We prove its contrapositive, which is logically the same: if two equivalence classes are not disjoint, then they are identical. Suppose that the equivalence classes $[a]$ and $[b]$ are not disjoint. This means there is a c in A such that $c \in [a]$ and $c \in [b]$. Hence

$$\begin{array}{lll} c \sim a & \text{and} & c \sim b \\ a \sim c & \text{and} & c \sim b & \text{(by symmetry)} \\ (1) & a \sim b & \text{(by transitivity)} \\ (2) & b \sim a & \text{(by symmetry).} \end{array}$$

We want to prove $[a] = [b]$. To this end, we have to prove $[a] \subseteq [b]$ and also $[b] \subseteq [a]$. Let us prove $[a] \subseteq [b]$. If $x \in [a]$, then $x \sim a$ and $a \sim b$ by (1), then $x \sim b$ by transitivity, then $x \in [b]$, so $[a] \subseteq [b]$. Similarly, if $y \in [b]$, then $y \sim b$, then $y \sim b$ and $b \sim a$ by (2), then $y \sim a$ by transitivity, then $y \in [a]$, so $[b] \subseteq [a]$. Hence $[a] = [b]$ if $[a]$ and $[b]$ are not disjoint. This completes the proof of the first assertion.

Now the converse. Let $A = \bigcup_{a \in A} P_i$ where any two distinct P_i 's are disjoint. We want to define an equivalence relation on A and want the P_i 's to be the equivalence classes. How do we accomplish this? Well, if the P_i are to be the equivalence classes, we had better call two elements equivalent if they belong to one and the same P_{i_0} .

Let $a \in A$. Since $A = \bigcup_{a \in A} P_i$, we see that $a \in P_{i_0}$ for some $i_0 \in I$. This index i_0 is uniquely determined by a . That is to say, a cannot belong to

two or more of the subsets P_i , for then P_i would not be mutually disjoint. So each element of A belongs to one and only one of the subsets P_i .

Let a, b be elements of A and suppose $a \in P_{i_0}$ and $b \in P_{i_1}$. We put $a \approx b$ if $P_{i_0} = P_{i_1}$, i.e., we put $a \approx b$ if the sets P_i to which a and b belong are identical. We show that \approx is an equivalence relation.

(i) For any $a \in A$, of course a belongs to the set P_{i_0} it belongs to, and so $a \approx a$ and \approx is reflexive.

(ii) Let $a \approx b$. This means a and b belong to the same set P_{i_0} , say, so b and a belong to the same set P_{i_0} , hence $b \approx a$. So \approx is symmetric.

(iii) Let $a \approx b$ and $b \approx c$. Then the set P_i to which b belongs contains a and c . Thus a and c belong to the same set P_i and $a \approx c$. This proves that \approx is transitive.

We showed that \approx is indeed an equivalence relation on A . It remains to prove that P_i are the equivalence classes under \approx . For any $a \in A$, we have, if $a \in P_{i_1}$,

$$\begin{aligned} [a] &= \{x \in A : x \approx a\} \\ &= \{x \in A : x \text{ belongs to } P_{i_1}\} \\ &= \{x \in \bigcup_{i \in I} P_i : x \text{ belongs to } P_{i_1}\} \\ &= P_{i_1}. \end{aligned}$$

This proves that P_i are the equivalence classes under \approx . □

Exercises

1. On $\mathbb{N} \times \mathbb{N}$, define a relation \equiv by declaring $(a, b) \equiv (c, d)$ if and only if $a + d = b + c$. Show that \equiv is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

2. Determine whether the relation \sim on \mathbb{R} is an equivalence relation on \mathbb{R} , when \sim is defined by declaring $x \sim y$ for all $x, y \in \mathbb{R}$ if and only if

- (a) there are integers a, b, c, d such that $ad - bc = \pm 1$ and $x = \frac{ay+b}{cy+d}$;
- (b) $|x - y| < 0.000001$;
- (c) $|x| = |y|$;
- (d) $x - y$ is an integer;
- (e) $x - y$ is an even integer;
- (f) there are natural numbers n, m such that $x^n = y^m$;
- (g) there are natural numbers n, m such that $nx = my$;
- (h) $x \geq y$.

3. Let \sim and \approx be two equivalence relations on a set A . We define \equiv by declaring $a \equiv b$ if and only if $a \sim b$ and $a \approx b$; and we define \cong by declaring $a \cong b$ if and only if $a \sim b$ or $a \approx b$. Determine whether \equiv and \cong are equivalence relations on A .

4. If a relation on A is symmetric and transitive, then it is also reflexive. Indeed, let \sim be the relation and let $a \in A$. Choose an element $b \in A$ such that $a \sim b$. Then $b \sim a$ by symmetry, and from $a \sim b$, $b \sim a$, it follows that $a \sim a$, by transitivity. So $a \sim a$ for any $a \in A$ and \sim is reflexive.

This argument is wrong. Why?