§4
Mathematical Induction

Examine the propositions

$$2^n \geqslant n \qquad \text{for all } n \in \mathbb{N},$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad \text{for all } n \in \mathbb{N},$$

$$n^{n+1} \geqslant (n+1)^n \qquad \text{for all } n \in \mathbb{N}.$$

How do we prove them? They are statements involving a variable n running through the infinite set $\mathbb{N}$. Strictly speaking, each one of these propositions above is a collection of infinitely many propositions. We can verify them for a finite number of cases where n assumes some specific values. Thus we might verify $2^n \geqslant n$ for $n = 1,2,3, \ldots ,1\,000\,000$ and convince ourselves of the truth of this statement, but this is far from a proof. On the other hand, we cannot check the truth of infinitely many statements within finite time. So we must resort to some other means.

In order to prove propositions about all natural numbers, an axiom is introduced. It is the fifth Peano axiom about $\mathbb{N}$ (Giuseppe Peano (1858-1932), an Italian mathematician and logician). It is called the axiom of mathematical induction.

4.1 Axiom (of mathematical induction): If S is a subset of $\mathbb{N}$ such that

I. $1 \in S$,

II. for all $k \in \mathbb{N}$, if $k \in S$, then $k + 1 \in \mathbb{N}$,

then S is the whole of $\mathbb{N}$, i.e., $S = \mathbb{N}$.

We can use this axiom to prove statements of the form 'p_n for all $n \in \mathbb{N}$' as follows. We let $S \subseteq \mathbb{N}$ be the set of all natural numbers n for which p_n is true. First we verify $1 \in S$, that is, we verify that p_1 is true. Second, we *assume* that $k \in S$ and under this hypothesis, which is called the *induction hypothesis*, we prove that p_{k+1} is true. So we show that $k \in S$ implies $k+1 \in S$. By the axiom of mathematical induction, $S = \mathbb{N}$, so the statement p_n is true for all $n \in \mathbb{N}$. We formulate the axiom as an operational procedure.

4.2 Principle of mathematical induction: Let p_n be a statement involving a natural number n. We can prove the proposition

$$\text{for all } n \in \mathbb{N},\ p_n$$

by establishing that

I. p_1 is true,

II. for all $k \in \mathbb{N}$, if p_k is true, then p_{k+1} is true.

Proofs by the principle of mathematical induction consist of two steps. In the first step, we show that p_1 is true. In practice, this is often quite easy, but we should not neglect it. In the second step, we assume that p_k is true. This assumption is the inductive hypothesis. Using this hypothesis, we prove that p_{k+1} is true. A proof by induction will not be complete (and valid) if we carry out the first step but not the second, or if we carry out the second step but not the first.

4.3 Examples: (a) Prove that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

We use the principle of mathematical induction.

I. $1 = \dfrac{1(1+1)}{2}$, so the formula is true for $n = 1$.

II. Make the inductive hypothesis that $1 + 2 + \cdots + k = \dfrac{k(k+1)}{2}$.

We want to establish $1 + 2 + \cdots + k + (k+1) = \dfrac{(k+1)((k+1)+1)}{2}$. We have

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) \quad \text{(by inductive hyp.)}$$

$$= \left(\frac{k}{2} + 1\right)(k+1)$$

$$= \frac{(k+1)(k+2)}{2},$$

so the formula is true for $n = k + 1$ if it is true for $n = k$. Hence

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}.$$

(b) Prove that $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$ for all $n \in \mathbb{N}$.

I. We have $2 = 2^{1+1} - 2$, which proves the assertion for $n = 1$.

II. Assume $2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$. Now we must prove $2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$. We have

$$\begin{aligned} 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} &= (2^{k+1} - 2) + 2^{k+1} \quad \text{(by inductive hyp.)} \\ &= 2(2^{k+1}) - 2 \\ &= 2^{k+2} - 2, \end{aligned}$$

so the assertion is true for $n = k + 1$ if it is true for $n = k$. Thus

$$2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2 \text{ for all } n \in \mathbb{N}.$$

(c) Let $h > -1$ be a fixed real number. Prove that $(1 + h)^n \geqslant 1 + nh$ for all $n \in \mathbb{N}$.

I. We have $(1 + h)^1 \geqslant 1 + 1h$, so the inequality is true for $n = 1$.

II. Let us assume $(1 + h)^k \geqslant 1 + kh$. We want to prove that $(1 + h)^{k+1} \geqslant 1 + (k + 1)h$. We have

$$\begin{aligned} (1 + h)^{k+1} &= (1 + h)^k (1 + h) \\ &\geqslant (1 + kh)(1 + h) \quad \text{(by inductive hyp. and } 1 + h \geqslant 0) \\ &= 1 + h + kh + kh^2 \\ &\geqslant 1 + h + kh + 0 \\ &= 1 + (k + 1)h, \end{aligned}$$

so the inequality is true for $n = k + 1$ if it is true for $n = k$. By the principle of mathematical induction,

$$(1 + h)^n \geqslant 1 + nh \text{ for all } n \in \mathbb{N}.$$

Sometimes it is convenient to use the principle of mathematical induction in a slightly different form. We assume (not only q_k, but rather) each one of $q_1, q_2, q_3, \ldots, q_k$ is true and then conclude that q_{k+1} is true. This establishes the truth of q_n for all $n \in \mathbb{N}$, as the following lemma shows.

4.4 Lemma: *Let q_n be a statement involving a natural number n. Assume that*

i. *q_1 is true,*

ii. *for all $k \in \mathbb{N}$, if $q_1, q_2, q_3, \ldots, q_k$ are true, then q_{k+1} is true.*

Then q_n is true for all $n \in \mathbb{N}$.

Proof: We prove the lemma by the principle of mathematical induction. We put

$$p_1 = q_1$$
$$p_k = q_1 \text{ and } q_2 \text{ and } \ldots \text{ and } q_k \qquad (\text{for all } k \in \mathbb{N}, k \geqslant 2).$$

Now induction.

I. p_1 is true (by the hypothesis i.)

II. Make the inductive hypothesis that p_k is true. Then

q_1 and q_2 and ... and q_k is true (definition of p_k)

$q_1, q_2, \ldots, q_k$ are all true (truth value of conjunction)

q_{k+1} is true (by the hypothesis ii.)

$q_1, q_2, \ldots, q_k, q_{k+1}$ are all true

q_1 and q_2 and ... and q_k and q_{k+1} is true

p_{k+1} is true.

Hence, for all $k \in \mathbb{N}$, if p_k is true, then p_{k+1} is true. By the principle of mathematical induction, p_n is true for all $n \in \mathbb{N}$. So

q_1 and q_2 and ... and q_n is true for all $n \in \mathbb{N}$.

In particular, q_n is true for all $n \in \mathbb{N}$. This completes the proof. □

We can now formulate a new form of the principle of mathematical induction. This form will be used many times in the sequel.

4.5 Principle of mathematical induction: Let q_n be a statement involving a natural number n. We can prove the proposition

for all $n \in \mathbb{N}$, q_n

by establishing that

I. q_1 is true,

II. for all $k \in \mathbb{N}$, if $q_1, q_2, \ldots, q_k$ are true, then q_{k+1} is true.

The statement '$2^n \geqslant n^2$' is not true for all natural numbers n, but true for all natural numbers $n \geqslant 5$. The principle of mathematical induction can be used to prove this and similar propositions. Let a be a fixed integer (positive, negative or zero) and let p_n be a statement involving an integer $n \geqslant a$. We prove the truth of p_n for all $n \geqslant a$ by showing that

1. p_a is true
2. for all $k \geqslant a$, if p_k is true, then p_{k+1} is true.

This is easily seen when we put $q_n = p_{n+a-1}$ for $n \in \mathbb{N}$ and use Principle 4.2 with q_n in place of p_n. There is a similar modification of Principle 4.5.

Exercises

Prove the assertions in Ex. 1-6 for all $n \in \mathbb{N}$ by the principle of mathematical induction.

1. $1 + 3 + \cdots + (2n - 1) = n^2$.
2. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$.
3. $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$.
4. $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$.
5. $1^4 + 2^4 + \cdots + n^4 = \dfrac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$.
6. Prove that $2^n \geqslant n^2$ for all $n \geqslant 5$, $n \in \mathbb{N}$.

7. Prove that $n^3 + 3n^2 + 2n + 1 \geqslant 0$ for all $n \geqslant -2$, $n \in \mathbb{Z}$.

8. Prove that, for any $n \in \mathbb{N}$ and for any positive real numbers $a_1, a_2, \ldots, a_{2^n}$,

$$\sqrt[2^n]{a_1 a_2 \ldots a_{2^n}} \leqslant \frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n}.$$

9. Prove that, for any $n \in \mathbb{N}$ and for any positive real numbers $a_1, a_2, \ldots, a_n$,

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leqslant \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

(Hint: if $m \neq 2^n$, then choose n so that $2^{n-1} < m < 2^n$. Put

$b = (a_1 + a_2 + \ldots + a_m)/m$. Then use Ex. 8 with $a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_{2^n}$, where $a_{m+1} = \ldots = a_{2^n} = b$.)