

§13 Groups of Isometries

For any nonempty set X , the set S_X of all one-to-one mappings from X onto X is a group under the composition of mappings (Example 7.1(d)). In particular, if X happens to be the Euclidean plane E , then E is the set of all points in the plane and S_E is a group. We note that E is not merely an ordinary set of points. An important feature of E is that there is a measure of distance between the points of E . Among the mappings in S_E , we examine those functions which preserve the distance between any two points. Clearly, such functions will be more important than other ones in S_E , since such mappings respect an important structure of the Euclidean plane E .

We choose an arbitrary but fixed cartesian coordinate system on E . Each point P in E will then be represented by the ordered pair (x,y) of its coordinates. We will not distinguish between the point P and the ordered pair (x,y) . So we write $(x,y)\alpha$ in place of $P\alpha$, where $\alpha \in S_E$. The distance between two points P,Q in E is given by $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, if P and Q have coordinates $(x_1,y_1),(x_2,y_2)$, respectively. This distance will be denoted by $d(P,Q)$ or by $d((x_1,y_1),(x_2,y_2))$.

13.1 Definition: A mapping $\alpha \in S_E$ is called an *isometry (of E)* if

$$d(P\alpha,Q\alpha) = d(P,Q)$$

for any two points P,Q in E .

This word is derived from "isos" and "metron", meaning "equal" and "measure" in Greek. The set of all isometries of E will be denoted by $Isom E$. Since the identity mapping $\iota_E: E \rightarrow E$ is evidently an isometry, $Isom E$ is a nonempty subset of S_E . In fact, $Isom E \leq S_E$.

13.2 Theorem: $Isom E$ is a subgroup of S_E .

Proof: We must show that the product of two isometries and the inverse of an isometry are isometries (Lemma 9.2).

(i) Let $\alpha, \beta \in \text{Isom } E$. Then, for any two points P, Q in E

$$\begin{aligned} d(P\alpha\beta, Q\alpha\beta) &= d((P\alpha)\beta, (Q\alpha)\beta) \\ &= d(P\alpha, Q\alpha) && \text{(since } \beta \text{ is an isometry)} \\ &= d(P, Q) && \text{(since } \alpha \text{ is an isometry),} \end{aligned}$$

so $\alpha\beta \in \text{Isom } E$. Hence $\text{Isom } E$ is closed under multiplication.

(ii) Let $\alpha \in \text{Isom } E$ and let P, Q be any two points in E . Since $\alpha \in S_E$, there are uniquely determined points P', Q' in E such that $P'\alpha = P$, $Q'\alpha = Q$. Thus $P' = P\alpha^{-1}$, $Q' = Q\alpha^{-1}$. Then

$$\begin{aligned} d(P', Q') &= d(P'\alpha, Q'\alpha) && \text{(since } \alpha \text{ is an isometry)} \\ d(P\alpha^{-1}, Q\alpha^{-1}) &= d(P, Q) \\ \alpha^{-1} &\in \text{Isom } E. \end{aligned}$$

Hence $\text{Isom } E \leq S_E$. □

We examine some special types of isometries, namely translations, rotations and reflections.

Loosely speaking, a translation shifts every point of E by the same amount in the same direction. In more detail, a translation is a mapping which "moves" any point (x, y) in E by a units in the direction of the x -axis and by b units in the direction of the y -axis (the directions being reversed when a or b is negative). See Figure 1. The formal definition is as follows.

13.3 Definition: A mapping $E \rightarrow E$ is called a *translation* if there are two real numbers a, b such that

$$(x, y) \rightarrow (x + a, y + b)$$

for all points (x, y) in E under this mapping.

The translation $(x, y) \rightarrow (x + a, y + b)$ will be denoted by $\tau_{a, b}$.

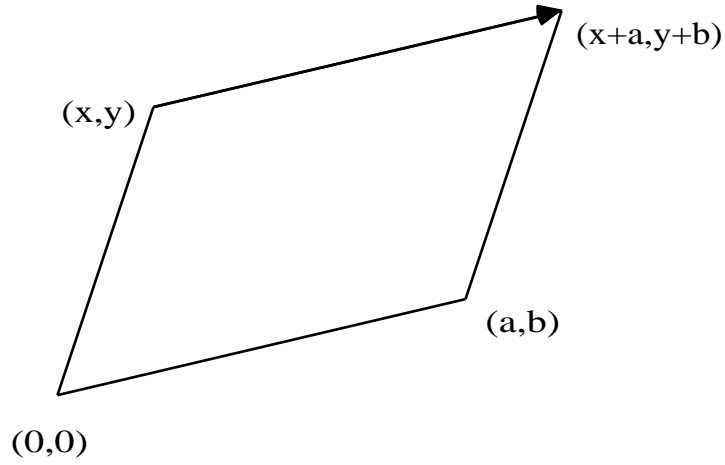


Figure 1

13.4 Lemma: Let $\tau_{a,b}$ and $\tau_{c,d}$ be arbitrary translations.

- (1) $\tau_{a,b}\tau_{c,d} = \tau_{a+c,b+d}$
- (2) $\tau_{0,0} = I_E = I$
- (3) $\tau_{-a,-b}\tau_{a,b} = I = \tau_{a,b}\tau_{-a,-b}$

Proof: (1) We have $(x,y)(\tau_{a,b}\tau_{c,d}) = ((x,y)\tau_{a,b})\tau_{c,d}$
 $= (x+a,y+b)\tau_{c,d}$
 $= ((x+a)+c,(y+b)+d)$
 $= (x+(a+c),y+(b+d))$
 $= (x,y)\tau_{a+c,b+d}$

for all $(x,y) \in E$. Thus $\tau_{a,b}\tau_{c,d} = \tau_{a+c,b+d}$.

(2) We have $(x,y)\tau_{0,0} = (x+0,y+0) = (x,y) = (x,y)I$ for all $(x,y) \in E$. Thus $\tau_{0,0} = I$.

(3) From (1) and (2) we get $\tau_{-a,-b}\tau_{a,b} = \tau_{(-a)+a,(-b)+b} = \tau_{0,0} = I$ and likewise $\tau_{a,b}\tau_{-a,-b} = \tau_{a+(-a),b+(-b)} = \tau_{0,0} = I$. \square

13.5 Lemma: Any translation is an isometry.

Proof: First of all, we must show that any translation belongs to S_E . Let $\tau_{a,b}$ be an arbitrary translation ($a,b \in \mathbb{R}$). There is a mapping $\psi: E \rightarrow E$ such that $\tau_{a,b}\psi = I = \psi\tau_{a,b}$, namely $\psi = \tau_{-a,-b}$ by Lemma 13.4(3). Thus $\tau_{a,b}$ is one-to-one and onto by Theorem 3.17(2). Hence $\tau_{a,b} \in S_E$.

Next we show $d((x_1,y_1),(x_2,y_2)) = d((x_1,y_1)\tau_{a,b},(x_2,y_2)\tau_{a,b})$ for any two points $(x_1,y_1),(x_2,y_2)$ in E . We have

$$\begin{aligned}
d((x_1, y_1)\tau_{a,b}, (x_2, y_2)\tau_{a,b}) &= d((x_1 + a, y_1 + b), (x_2 + a, y_2 + b)) \\
&= \sqrt{[(x_1 + a) - (x_2 + a)]^2 + [(y_1 + b) - (y_2 + b)]^2} \\
&= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= d((x_1, y_1), (x_2, y_2))
\end{aligned}$$

and so $\tau_{a,b} \in \text{Isom } E$. □

13.6 Theorem: *The set T of all translations is a subgroup of $\text{Isom } E$.*

Proof: Let $T = \{\tau_{a,b}; a, b \in \mathbb{R}\}$ be the set of all translations. T is a subset of $\text{Isom } E$ by Lemma 13.5. From Lemma 13.4(2), $t = \tau_{0,0} \in T$, so $T \neq \emptyset$. Now we use our subgroup criterion (Lemma 9.2).

(i) The product of two translations $\tau_{a,b}$ and $\tau_{c,d}$ is a translation $\tau_{a+c, b+d} \in T$ by Lemma 13.4(1). So T is closed under multiplication.

(ii) The inverse of any translation $\tau_{a,b} \in T$ is also a translation $\tau_{-a, -b} \in T$ by Lemma 13.4(3). So T is closed under taking inverses.

Thus T is a subgroup of $\text{Isom } E$. □

Next we investigate rotations. By a rotation about a point C through an angle φ , we want to understand a mapping from E into E which sends the point C to C and whose effect on any point $P \neq C$ is as follows: we turn the line segment CP about the point C through the angle φ into a new line segment, say to CQ ; the point P will be sent to the point Q (see Figure 2). We recall that positive values of φ measure counterclockwise angles and negative values of φ measure clockwise angles.

Rotations are most easily described in a polar coordinate system. We choose the center of rotation, the point C , as the pole. The initial ray is chosen arbitrarily. The point P with polar coordinates (r, θ) is then sent to the point Q whose polar coordinates are $(r, \theta + \varphi)$. If C is the origin and the initial ray is the positive x -axis of a cartesian coordinate system, then the polar and cartesian coordinates of a point P are connected by

$$x = r \cos \theta \qquad y = r \sin \theta.$$

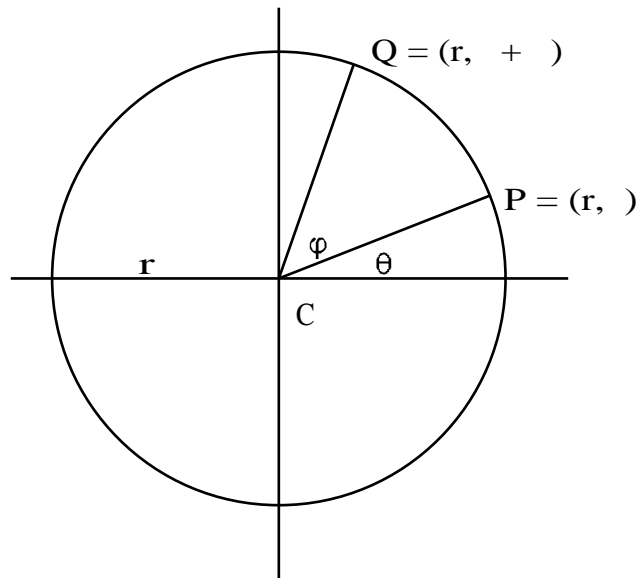


Figure 2

In our fixed cartesian coordinate system, the image of any point $P = (x,y)$ can be found as follows. If P has polar coordinates (r,θ) , then its image will have polar coordinates $(r,\theta + \varphi)$, so the cartesian coordinates x',y' of $Q := (r,\theta + \varphi)$ are

$$\begin{aligned} x' &= r \cos(\theta + \varphi) = r(\cos\theta \cos\varphi - \sin\theta \sin\varphi) \\ &= (r \cos\theta) \cos\varphi - (r \sin\theta) \sin\varphi \\ &= x \cos\varphi - y \sin\varphi, \\ y' &= r \sin(\theta + \varphi) = r(\sin\theta \cos\varphi + \cos\theta \sin\varphi) \\ &= (r \sin\theta) \cos\varphi + (r \cos\theta) \sin\varphi \\ &= y \cos\varphi + x \sin\varphi \\ &= x \sin\varphi + y \cos\varphi. \end{aligned}$$

This suggests the following formal definition.

13.7 Definition: A mapping $E \rightarrow E$ is called a *rotation about the origin through an angle φ* if there is a real number φ such that

$$(x,y) \rightarrow (x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi)$$

for all points (x,y) in E under this mapping.

The rotation $(x,y) \rightarrow (x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi)$ will be denoted by ρ_φ . We have an analogue of Lemma 13.4 for rotations.

13.8 Lemma: Let ρ_φ and ρ_ψ be arbitrary rotations about the origin.

(1) $\rho_\varphi\rho_\psi = \rho_{\varphi+\psi}$.

(2) $\rho_0 = I_E = I$.

(3) $\rho_{-\varphi}\rho_\varphi = I = \rho_\varphi\rho_{-\varphi}$.

Proof: (1) We have $(x,y)(\rho_\varphi\rho_\psi) = ((x,y)\rho_\varphi)\rho_\psi$
 $= (x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi)\rho_\psi$
 $= ((x \cos\varphi - y \sin\varphi)\cos\psi - (x \sin\varphi + y \cos\varphi)\sin\psi, (x \cos\varphi - y \sin\varphi)\sin\psi + (x \sin\varphi + y \cos\varphi)\cos\psi)$
 $= (x(\cos\varphi \cos\psi - \sin\varphi \sin\psi) - y(\sin\varphi \cos\psi + \cos\varphi \sin\psi),$
 $\quad x(\cos\varphi \sin\psi + \sin\varphi \cos\psi) + y(-\sin\varphi \sin\psi + \cos\varphi \cos\psi))$
 $= (x \cos(\varphi + \psi) - y \sin(\varphi + \psi), x \sin(\varphi + \psi) + y \cos(\varphi + \psi))$
 $= (x,y)\rho_{\varphi+\psi}$
for all $(x,y) \in E$. Thus $\rho_\varphi\rho_\psi = \rho_{\varphi+\psi}$.

(2) We have $(x,y)\rho_0 = (x \cos 0 - y \sin 0, x \sin 0 + y \cos 0) = (x - 0, 0 + y)$
 $= (x,y) = (x,y)I$
for all $(x,y) \in E$. Thus $\rho_0 = I$.

(3) From (1) and (2) we get $\rho_{-\varphi}\rho_\varphi = \rho_{(-\varphi)+\varphi} = \rho_0 = I$ and likewise $\rho_\varphi\rho_{-\varphi} = \rho_{\varphi+(-\varphi)} = \rho_0 = I$. □

Lemma 13.8 was to be expected. When we carry out a rotation through an angle φ and then a rotation through an angle ψ , we have in effect a rotation through an angle $\varphi + \psi$. This is what Lemma 13.8(1) states. Also, when we carry out a rotation through an angle φ and then a rotation through the same angle in the reverse direction, the final result will be: no net motion at all. This is what Lemma 13.8(3) states.

13.9 Lemma: Any rotation about the origin is an isometry.

Proof: First of all, we must show that any rotation about the origin belongs to S_E . Let ρ_φ be an arbitrary rotation about the origin ($\varphi \in \mathbb{R}$). There is a mapping $\psi: E \rightarrow E$ such that $\rho_\varphi\psi = I = \psi\rho_\varphi$, namely $\psi = \rho_{-\varphi}$ by Lemma 13.8(3). Thus ρ_φ is one-to-one and onto by Theorem 3.17(2). Hence $\rho_\varphi \in S_E$.

Now we prove that ρ_φ preserves distance. For any two points $(x,y),(u,v)$ in E , we have

$$\begin{aligned}
& d^2((x,y)\rho_\varphi,(u,v)\rho_\varphi) \\
&= d^2((x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi),(u \cos\varphi - v \sin\varphi, u \sin\varphi + v \cos\varphi)) \\
&= [(x - u)\cos\varphi - (y - v)\sin\varphi]^2 + [(x - u)\sin\varphi + (y - v)\cos\varphi]^2 \\
&= (x - u)^2\cos^2\varphi - 2(x - u)(y - v)\cos\varphi \sin\varphi + (y - v)^2\sin^2\varphi \\
&\quad + (x - u)^2\sin^2\varphi + 2(x - u)(y - v)\cos\varphi \sin\varphi + (y - v)^2\cos^2\varphi \\
&= (x - u)^2(\cos^2\varphi + \sin^2\varphi) + (y - v)^2(\sin^2\varphi + \cos^2\varphi) \\
&= (x - u)^2 + (y - v)^2 \\
&= d^2((x,y),(u,v)),
\end{aligned}$$

hence $d((x,y)\rho_\varphi,(u,v)\rho_\varphi) = d((x,y),(u,v))$. So ρ_φ is an isometry. \square

13.10 Theorem: *The set R of all rotations about the origin is a subgroup of $Isom E$.*

Proof: Let $R = \{\rho_\varphi : \varphi \in \mathbb{R}\}$ be the set of all rotations about the origin. R is a subset of $Isom E$ by Lemma 13.9. By Lemma 13.8(2), $\iota = \rho_0 \in R$, so $R \neq \emptyset$. Now we use our subgroup criterion (Lemma 9.2).

(i) The product of two rotations ρ_φ and ρ_ψ about the origin is a rotation $\rho_{\varphi+\psi} \in R$ about the origin by Lemma 13.8(1). So R is closed under multiplication.

(ii) The inverse of any rotation $\rho_\varphi \in R$ about the origin is also a rotation $\rho_{-\varphi} \in R$ about the origin by Lemma 13.8(3). So R is closed under taking inverses.

Thus R is a subgroup of $Isom E$. \square

So far, we have been dealing with rotations about the origin. What about rotations about an arbitrary point C , whose coordinates are (a,b) , say. A rotation about C through an angle φ will map a point P with coordinates $(x + a, y + b)$ to a point Q with coordinates $(x' + a, y' + b)$, where (x', y') is the point to which (x, y) is mapped under a rotation about the origin through an angle φ . So the image of $(x, y)\tau_{a,b}$ will be $(x, y)\rho_\varphi\tau_{a,b}$. See Figure 3. This suggests the following formal definition.

13.11 Definition: Let $C = (a,b)$ be a point in E . The mapping $(\tau_{a,b})^{-1}\rho_\varphi\tau_{a,b}: E \rightarrow E$ is called a *rotation about C through an angle φ* .

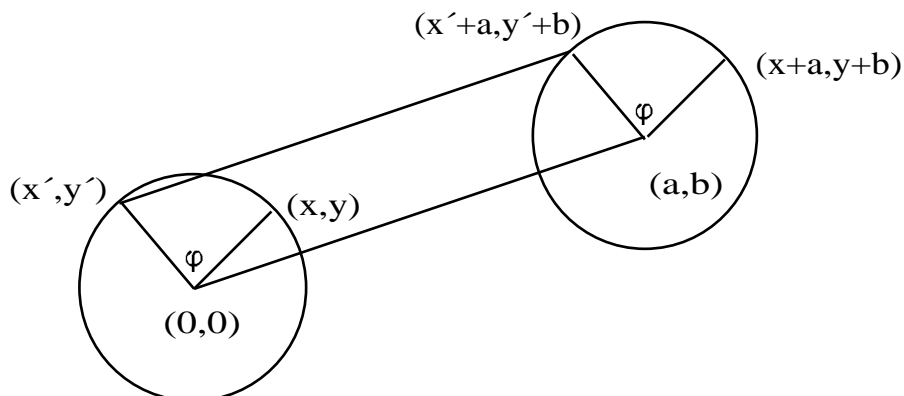


Figure 3

We put $(\tau_{a,b})^{-1}R\tau_{a,b} := \{(\tau_{a,b})^{-1}\rho_\varphi\tau_{a,b} : \rho_\varphi \in R\}$. This is the set of all rotations about the point (a,b) . It is a subgroup of $Isom E$. The proof of this statement is left to the reader.

Now we examine reflections. The cartesian equations of a reflection are very cumbersome. For this reason, we give a coordinate-free definition of reflections. We need some notation. Let P, Q be distinct points in the plane E . In what follows, \overline{PQ} will denote the line through P and Q , and \underline{PQ} will denote the line segment between P and Q . So \overline{PQ} is the set of points R in E such that $d(P,R) + d(R,Q) = d(P,Q)$.

The geometric idea of a reflection is that there is a line m and that each point P is mapped to its "mirror image" Q on the other side of m . So \overline{PQ} is perpendicular to m and $d(P,R) = d(R,Q)$, where R is the point of intersection of m and \overline{PQ} . See Figure 4.

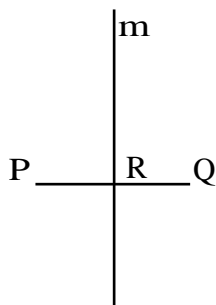


Figure 4

13.12 Definition: Let m be a straight line in E and let $\sigma_m: E \rightarrow E$ be the mapping defined by

$$P\sigma_m = P \text{ if } P \text{ is on } m$$

and

$P\sigma_m = Q$ if P is not on m and if m is the perpendicular bisector of \overline{PQ} . σ_m is called the *reflection in the line m* .

The perpendicular bisector of \overline{PQ} is the line that is perpendicular to \overline{PQ} and that intersects \overline{PQ} at a point R such that $d(P,R) = d(R,Q)$. It is also the locus of all points in E which are equidistant from P and Q . So it is the set $\{R \in E: d(P,R) = d(Q,R)\}$. We will make use of this description of the perpendicular bisector in the sequel without explicit mention.

13.13 Lemma: Let σ_m be the reflection in a line m . Then $\sigma_m \neq \iota = \sigma_m^2$.

Proof: $P\sigma_m \neq P$ if P is not on the line m and so $\sigma_m \neq \iota$. Now we prove that $\sigma_m^2 = \iota$. We have $P\sigma_m^2 = P(\sigma_m\sigma_m) = (P\sigma_m)\sigma_m = P\sigma_m = P$ when P is a point on m by definition. It remains to show $P\sigma_m^2 = P$ also when P is not on m . Let P be a point not on m and let $Q = P\sigma_m, P_1 = Q\sigma_m$. Then \overline{PQ} is not on m and m is the perpendicular bisector of \overline{PQ} as well as of $\overline{QP_1}$. So \overline{PQ} and $\overline{QP_1}$ are parallel lines. Since they have a point Q in common, they are identical lines. Let R be the point at which m and \overline{PQ} intersect. So $\overline{P_1} \neq Q$ and P_1 is that point on \overline{PQ} for which $d(Q,R) = d(P_1,R)$. Since P is on \overline{PQ} and $d(Q,R) = d(P,R)$, we obtain $P = P_1$, as was to be proved. \square

13.14 Lemma: Any reflection in a line is an isometry.

Proof: Let m be a line, σ_m the reflection in m and let P, Q be arbitrary points in the plane. We put $P_1 = P\sigma_m, Q_1 = Q\sigma_m$. We are to show $d(P,Q) = d(P_1,Q_1)$. We distinguish several cases.

Case 1. Assume both P and Q are on m . Then $P_1 = P$ and $Q_1 = Q$. So $d(P,Q) = d(P_1,Q_1)$.

Case 2. Assume one of the points is on m , the other is not. We suppose, without loss of generality, that P is on m and Q is not on m . Let Q_1

intersect m at S . Then $d(Q,S) = d(S,Q_1)$, $d(P,S) = d(S,P_1)$ since $P = P_1$ and the angles $\angle PSQ$ and $\angle P_1Q_1S$ are both right angles. By the side-angle-side condition, the triangles $\triangle QPS$ and $\triangle Q_1P_1S$ are congruent. So the corresponding sides PQ and P_1Q_1 have equal length. This means $d(P,Q) = d(P_1,Q_1)$.

From now on, assume that neither P nor Q is on m . Let m intersect $\overline{PP_1}$ at N and $\overline{QQ_1}$ at S .

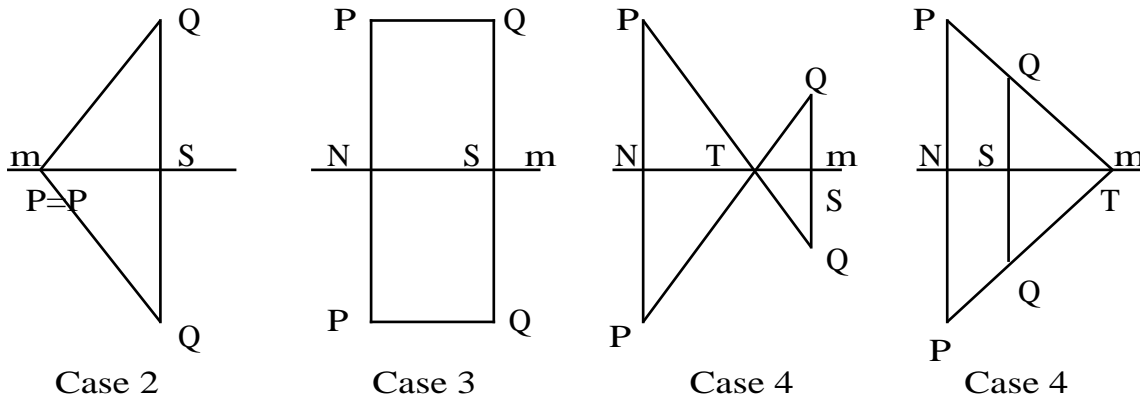


Figure 5

Case 3. Assume that \overline{PQ} is parallel to m . Then the quadrilaterals $\square NPQS$ and $\square NP_1Q_1S$ are rectangles. So $\square PP_1Q_1Q$ is a rectangle and the sides PQ and P_1Q_1 have equal length. This means $d(P,Q) = d(P_1,Q_1)$.

Case 4. Assume that \overline{PQ} is not parallel to m . Then \overline{PQ} intersects m at a point T . As in case 2, $\triangle PTN$ and $\triangle P_1TN$ are congruent, and $\triangle QST$ and $\triangle Q_1ST$ are congruent, so $d(P,T) = d(P_1,T)$ and $d(Q,T) = d(Q_1,T)$. Also, $\angle STQ_1 = \angle STQ = \angle NTP = \angle NTP_1$, which shows that P_1, T, Q_1 lie on a straight line. Then we obtain

$$\begin{aligned}
 d(P_1, Q_1) &= d(P_1, T) \mp d(T, Q_1) \\
 &= d(P_1, T) \mp d(Q_1, T) \\
 &= d(P, T) \mp d(Q, T) \\
 &= d(P, T) \mp d(T, Q) \\
 &= d(P, Q),
 \end{aligned}$$

where the upper or lower sign is to be taken according as whether P, Q are on the same or on the opposite sides of m . □

13.15 Theorem: Let m be a line in E . Then $\{1, \sigma_m\}$ is a subgroup of $\text{Isom } E$.

Proof: $\{t, \sigma_m\}$ is a finite nonempty subset of $Isom E$ by Lemma 13.14. It is closed under multiplication by Lemma 13.13. So it is a subgroup of $Isom E$ by Lemma 9.3(1).

□

Translations, rotations and reflections are isometries. Thus the products of any number of these mappings, carried out in any order, will be isometries, too. We show in the rest of this paragraph that all isometries are obtained in this way. We need some lemmas.

13.16 Lemma: *Let P, Q, R be arbitrary points in E .*

(1) *There is a unique translation that maps P to Q .*

(2) *If $d(P, Q) = d(P, R)$, there is a rotation about P that maps Q to R .*

Proof: (1) When $P = (a, b)$ and $Q = (c, d)$, say, then $\tau_{m, n}$ maps P to Q if and only if $(a + m, b + n) = (c, d)$, i.e., if and only if $m = c - a$, $n = d - b$. So $\tau_{c-a, d-b}$ is the unique translation that maps P to Q .

(2) We draw the circle whose center is at P and whose radius is equal to $d(P, Q)$. This circle passes through R by hypothesis. Let φ be the angle which the circular arc \widehat{QR} subtends at the center P . Then a rotation about P through an angle φ maps Q to R . □

The next lemma states that an isometry is completely determined by its effect on three points not lying on a line.

13.17 Lemma: *Let P, Q, R be three distinct points in E that do not lie on a straight line. Let α, β be isometries such that $P\alpha = P\beta$, $Q\alpha = Q\beta$, $R\alpha = R\beta$. Then $\alpha = \beta$.*

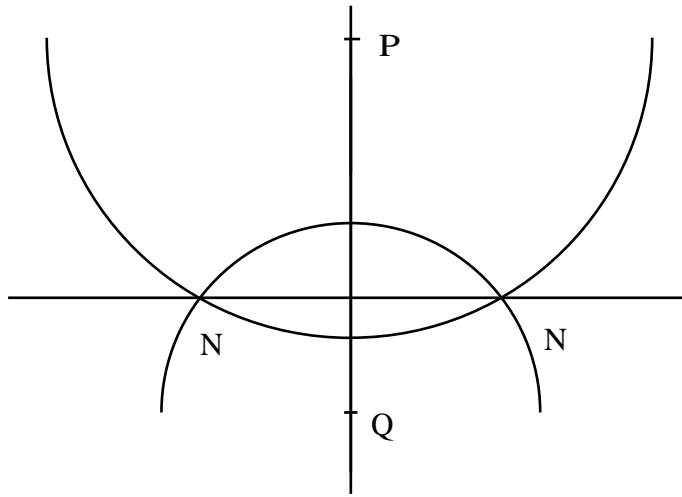


Figure 6

Proof: We put $\alpha\beta^{-1} = \gamma$. We suppose $\gamma \neq \iota$ and try to reach a contradiction. If $\gamma \neq \iota$, then there is a point N in E such that $N \neq N\gamma$. Since $P\alpha = P\beta$ by hypothesis, $P\gamma = P$ and so $P \neq N$. Similarly $Q \neq N$ and $R \neq N$. Now γ is an isometry, so $d(P, N\gamma) = d(P\gamma, N\gamma) = d(P, N)$ and likewise $d(Q, N\gamma) = d(Q, N)$ and $d(R, N\gamma) = d(R, N)$. So the circle with center at P and radius $d(P, N)$ and the circle with center at Q and radius $d(Q, N)$ intersect at the points N and $N\gamma$. Then \overline{PQ} is the perpendicular bisector of $\overline{N N\gamma}$. Here we used $N \neq N\gamma$. But $d(R, N\gamma) = d(R, N)$ and R lies therefore on the perpendicular bisector of $\overline{N N\gamma}$, i.e., R lies on \overline{PQ} , contrary to the hypothesis that P, Q, R do not lie on a straight line. Hence necessarily $\gamma = \iota$ and $\alpha = \beta$. \square

13.18 Theorem: Let P, Q, R be three distinct points in E that do not lie on a straight line and let P', Q', R' be three distinct points in E . Assume that $d(P, Q) = d(P', Q')$, $d(P, R) = d(P', R')$, $d(Q, R) = d(Q', R')$. Then there is a translation τ , a rotation ρ (about an appropriate point and through a suitable angle) and a reflection σ such that

$$P' = P\beta, Q' = Q\beta, R' = R\beta,$$

where β denotes $\tau\rho$ or $\tau\rho\sigma$.

Proof: By Lemma 13.16(1), there is a translation τ that maps P to P' . We put $Q_1 = Q\tau$ and $R_1 = R\tau$. Since τ is an isometry, $d(P, Q) = d(P\tau, Q\tau) = d(P', Q_1)$, so $d(P', Q_1) = d(P', Q')$. By Lemma 13.16(2), there is a rotation about P' that maps Q_1 to Q' . Let us denote this rotation by ρ . Then $P'\rho = P'$. We put

$$R_1\rho = R_2.$$

Here it may happen that $R_2 = R'$. Putting $\beta = \tau\rho$ in this case, we have $P' = P\beta$, $Q' = Q\beta$, $R' = R\beta$, as claimed.

Suppose now $R_2 \neq R'$. From $d(P',R') = d(P,R) = d(P\tau\rho,R\tau\rho) = d(P',R_2)$ and $d(Q',R') = d(Q,R) = d(Q\tau\rho,R\tau\rho) = d(Q',R_2)$, we deduce that both P' and Q' lie on the perpendicular bisector of $\overline{R'R_2}$. Denoting the reflection in the line $\overline{P'Q'}$ by σ , we get $P'\sigma = P'$, $Q'\sigma = Q'$ and $R_2\sigma = R'$. Putting $\beta = \tau\rho\sigma$ in this case, we have $P' = P\beta$, $Q' = Q\beta$, $R' = R\beta$, as claimed.

The proof is summarized schematically below.

$$\begin{array}{ccccccc}
 & & \tau & & \rho & & \sigma \\
 P & \rightarrow & P' & \rightarrow & P' & \rightarrow & P' \\
 Q & \rightarrow & Q_1 & \rightarrow & Q' & \rightarrow & Q' \\
 R & \rightarrow & R_1 & \rightarrow & R_2 & \rightarrow & R'
 \end{array}
 \quad \square$$

13.19 Theorem: *Every isometry can be written as a product of translations, rotations and reflections. In fact, if α is an arbitrary isometry, then there is a translation τ , a rotation ρ and a reflection σ such that*

$$\alpha = \tau\rho \text{ or } \tau\rho\sigma.$$

Proof: Let α be an arbitrary isometry. Choose any three distinct points P, Q, R in E not lying on a straight line. Then $d(P, Q) = d(P\alpha, Q\alpha)$, $d(P, R) = d(P\alpha, R\alpha)$, $d(Q, R) = d(Q\alpha, R\alpha)$. So the hypotheses of Theorem 13.18 are satisfied with $P' = P\alpha$, $Q' = Q\alpha$, $R' = R\alpha$ and there is a translation τ , a rotation ρ and a reflection σ such that

$$P\alpha = P\beta, Q\alpha = Q\beta, R\alpha = R\beta,$$

where $\beta = \tau\rho$ or $\tau\rho\sigma$. By Lemma 13.17, $\alpha = \beta$. Thus $\alpha = \tau\rho$ or $\tau\rho\sigma$.

□

Exercises

1. Let m be the line in E whose cartesian equation is $ax + by + c = 0$. Show that the reflection σ_m in the line m is given by

$$(u, v)\sigma_m = \left(u - \frac{2a}{a^2 + b^2}(au + bv + c), v - \frac{2b}{a^2 + b^2}(au + bv + c)\right).$$

2. Let m and n be two distinct lines intersecting at a point P . Show that $\sigma_m \sigma_n$ is a rotation about P . Through which angle?
3. Let m and n be parallel lines. Show that $\sigma_m \sigma_n$ is a translation.
4. Prove that every rotation and every translation can be written as a product of two reflections.
5. Prove that every isometry can be written as a product of reflections.
6. A *halfturn* $\sigma_P = \sigma_{(a,b)}$ about a point $P = (a,b)$ is defined as the mapping given by

$$(x,y) \rightarrow (2a - x, 2b - y)$$
 for all points (x,y) in E . Show that any halfturn is an isometry of order two. Prove that the product of three halfturns is a halfturn.
7. Prove that a halfturn σ_P is the product of any two reflections in lines intersecting perpendicularly at P .
8. Prove that a product of two halfturns is a translation.
9. Show that the set of all translations and halfturns is a subgroup of $Isom E$.
10. Prove that the product of four reflections can be written as a product of two reflections.
11. Show that $\rho_{2\pi/n}$ generates a cyclic subgroup of order n of $Isom E$.
12. Prove that every nonidentity translation is of infinite order and that ρ_φ is of finite order if and only if φ is a rational multiple of π .