§16
Alternating Groups

In this paragraph, we examine an important subgroup of S_n, called the alternating group on n letters. We begin with a definition that will play an important role throughout this paragraph.

16.1 Definition: A cycle of length 2 in S_n (where $n \geqslant 2$) is called a *transposition*.

A transposition is therefore a permutation of the form (ab) and has order 2 (Theorem 15.11). We remark that $(ab) = (ba)$.

16.2 Theorem: *Any permutation in* S_n (*where* $n \geqslant 2$) *can be written as a product of transpositions.*

Proof: Since any permutation in S_n can be written as a product of (disjoint) cycles (Theorem 15.9), it suffices to prove that any cycle can be written as a product of transpositions. This follows from $(abc\dots e) = (ab)(ac)\dots(ae)$ for cycles of length > 1. Also $\iota = (12)(12)$ is a product of transpositions. This completes the proof. □

There is no uniqueness claim in Theorem 16.2. A permutation can be written as a product of different transpositions. For instance,

$$(12345) = (12)(13)(14)(15) = (45)(41)(42)(43)$$

is written as a product of different transpositions. Nor is the number of transpositions is unique. The permutation (132546) can be written as a product of five or nine transpositions:

$$(132546) = (13)(12)(15)(14)(16) =$$
$$(24)(12)(14)(23)(46)(14)(16)(45)(16).$$

In fact, we can attach a product of two transpositions $(ab)(ab) = \iota$ at will and increase the number of transpositions by 2. Hence a product of n transpositions can be written also as a product of $n + 2, n + 4, n + 6, \ldots$ transpositions. We note that this does not change the *parity* of the number of transpositions. The parity of the number of transpositions is unique. If a permutation can be written as a product of an odd (even) number of transpositions, then, in any representation of this permutation as a product of transpositions, the number of transpositions is odd (even). A permutation cannot be written as a product of an odd number of transpositions and also as a product of an even number of transpositions. We proceed to prove this assertion. We need the notion of inversions of a permutation.

Let $\sigma \in S_n$. We write σ in double row notation, where, in the first row, the numbers $1,2, \ldots,n$ are in their natural order:

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ 1\sigma & 2\sigma & \ldots & n\sigma \end{pmatrix}.$$

Corresponding to the correct inequalities

$$\begin{array}{cccc} 1 < 2 & 1 < 3 & \ldots\ldots & 1 < n \\ & 2 < 3 & \ldots\ldots & 2 < n \\ & & \ldots\ldots\ldots\ldots & \\ & & & n-1 < n \end{array}$$

among the numbers in the first row, we obtain the inequalities

$$\begin{array}{cccc} 1\sigma < 2\sigma & 1\sigma < 3\sigma & \ldots\ldots & 1\sigma < n\sigma \\ & 2\sigma < 3\sigma & \ldots\ldots & 2\sigma < n\sigma \\ & & \ldots\ldots\ldots\ldots & \\ & & & (n-1)\sigma < n\sigma \end{array}$$

among the numbers in the second row when we replace each k by $k\sigma$ ($k = 1,2, \ldots,n$). These inequalities will be referred to as the inequalities of σ. In general, some of the inequalities of σ will be correct, some will be wrong (if $\sigma \neq \iota$, there will be a wrong inequality of σ). A wrong inequality $i\sigma < j\sigma$ of σ means: $i < j$ but $i\sigma > j\sigma$ i.e., the natural order of i and j is inverted in the second row (that is, the larger one precedes the smaller one). We call each wrong inequality of σ an *inversion of* σ. For example, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 3 & 1 & 4 \end{pmatrix}$ has the inequalities

$2 < 5$	$2 < 6$	$2 < 3$	$2 < 1$	$2 < 4$
	$5 < 6$	$5 < 3$	$5 < 1$	$5 < 4$
		$6 < 3$	$6 < 1$	$6 < 4$
			$3 < 1$	$3 < 4$
				$1 < 4$,

eight of which are wrong, namely $2 < 1$, $5 < 3$, $5 < 1$, $5 < 4$, $6 < 3$,
$6 < 1$, $6 < 4$, $3 < 1$. Hence there are eight inversions of σ.

The main work of this paragraph is done in the next lemma.

16.3 Lemma: *Let $n \geqslant 2$, $\sigma \in S_n$ and let (ik) be a transposition in S_n. If σ has an odd number of inversions, then $(ik)\sigma$ has an even number of inversions. If σ has an even number of inversions, then $(ik)\sigma$ has an odd number of inversions.*

Proof: Since $(ik) = (ki)$, we assume, without loss of generality, that $i < k$. We have

$$\sigma = \begin{pmatrix} 1 & \dots & i & \dots & k & \dots & n \\ 1\sigma & \dots & i\sigma & \dots & k\sigma & \dots & n\sigma \end{pmatrix}, \quad (ik)\sigma = \begin{pmatrix} 1 & \dots & i & \dots & k & \dots & n \\ 1\sigma & \dots & k\sigma & \dots & i\sigma & \dots & n\sigma \end{pmatrix}.$$

The second rows of σ and $(ik)\sigma$ are identical, aside from the locations of $i\sigma$ and $k\sigma$. Here σ gives rise to the inequalities

1.	$h\sigma < i\sigma$,	$h\sigma < k\sigma$	where $h \in \{1, \dots, i-1\} =: \mathrm{H}$,
	$i\sigma < j\sigma$,		where $j \in \{i+1, \dots, k-1\} =: \mathrm{J}$,
	$i\sigma < k\sigma$,		
2.	$i\sigma < m\sigma$,		where $m \in \{k+1, \dots, n\} =: \mathrm{M}$,
	$j\sigma < k\sigma$,		where $j \in \mathrm{J}$,
3.	$k\sigma < m\sigma$,		where $m \in \mathrm{M}$,

and to certain other inequalities that do not involve $i\sigma$ or $k\sigma$. And $(ik)\sigma$ gives rise to the inequalities

1.	$h\sigma < k\sigma$,	$h\sigma < i\sigma$	where $h \in \mathrm{H}$,
	$k\sigma < j\sigma$,		where $j \in \mathrm{J}$,
	$k\sigma < i\sigma$,		
3.	$k\sigma < m\sigma$,		where $m \in \mathrm{M}$,
	$j\sigma < i\sigma$,		where $j \in \mathrm{J}$,
2.	$i\sigma < m\sigma$,		where $m \in \mathrm{M}$,

and to certain other inequalities that do not involve $i\sigma$ or $k\sigma$.

In the cases $i = 1$, $k = i + 1$, $k = n$, there holds respectively H = ∅, J = ∅, M = ∅ and the correponding inequalities should be deleted. This does not impair the argument below.

We are to show that the number of inversions of σ and the number of inversions of $(ik)\sigma$ differ by an odd number.

The inequalities of σ and of $(ik)\sigma$ that do not involve $i\sigma$ or $k\sigma$ are identical. Also, the inequalities 1., 2., 3. of σ and $(ik)\sigma$ are the same (or absent). So only the inequalities

I. $i\sigma < j\sigma$, $i\sigma < k\sigma$, $j\sigma < k\sigma$ (where $j \in$ J) of σ

and

II. $k\sigma < j\sigma$, $k\sigma < i\sigma$, $j\sigma < i\sigma$ (where $j \in$ J) of $(ik)\sigma$

are different. We must prove that the number of wrong inequalities in I and II differ by an odd number.

Since one of $i\sigma < k\sigma$, $k\sigma < i\sigma$ is correct and the other is wrong, we must prove only that the number of wrong inequalities in

A. $i\sigma < j\sigma$, $j\sigma < k\sigma$ (where $j \in$ J)

and in

B. $k\sigma < j\sigma$, $j\sigma < i\sigma$ (where $j \in$ J)

differ by an even number.

Suppose there are s wrong inequalities $i\sigma < j\sigma$ and t wrong inequalities $j\sigma < k\sigma$ in A, where $|J| \geqslant s \geqslant 0$ and $|J| \geqslant t \geqslant 0$ (including the case J = ∅, $|J| = 0$). Then there are $s + t$ wrong inequalities and there are $(|J| - s) + (|J| - t) = 2|J| - (s + t)$ correct inequalities in A. Since B consists of the negations of the inequalities in A, there are $2|J| - (s + t)$ wrong inequalities in B. So

$$\begin{aligned}&(\text{no of wrong inequalities in A}) - (\text{no of wrong inequalities in B})\\ &\qquad = (s + t) - \big(2|J| - (s + t)\big) = 2(s + t - |J|) = \text{an even number.}\end{aligned}$$

This completes the proof. □

16.4 Definition: Let $n \in \mathbb{N}$ and let $\sigma \in S_n$. If σ has an odd number of inversions, then σ is called an *odd permutation*. If σ has an even number of inversions, then σ is called an *even permutation*.

As the number of inversions of a permutation is uniquely determined, it is clear that a permutation cannot be both odd and even. With this terminology, Lemma 16.3 reads as follows.

16.3 Lemma: *Let $n \geqslant 2$ and $\sigma \in S_n$. Let (ik) be a transposition in S_n. If σ is odd, then $(ik)\sigma$ is even. If σ is even, then $(ik)\sigma$ is odd.* □

Applying Lemma 16.3 r times, we have

16.5 Lemma: *Let $n \geqslant 2$, $\sigma \in S_n$ and let $\tau_1, \tau_2, \ldots, \tau_r$ be transpositions in S_n. If r is odd, then σ and $\tau_1\tau_2 \ldots \tau_r\sigma$ have the opposite "parity" (i.e., one of them is odd, the other is even). If r is even, then σ and $\tau_1\tau_2 \ldots \tau_r\sigma$ have the same "parity".* □

16.6 Theorem: *Let $n \geqslant 2$, $\pi \in S_n$. Then π is an odd (even) permutation if and only if π can be written as a product of an odd (even) number of transpositions. In particular, π cannot be written as a product of an odd number of transpositions and also as a product of an even number of transpositions.*

Proof: We use Lemma 16.5 with $\sigma = \iota$. Let π be written as a product of transpositions, say $\pi = \tau_1\tau_2 \ldots \tau_r$. Lemma 16.5 tells us that $\pi = \tau_1\tau_2 \ldots \tau_r\iota$ and ι have opposite or same "parities" according as whether r is odd or even. Since ι has 0 inversions, ι is an even permutation. So $\pi = \tau_1\tau_2 \ldots \tau_r$ is an odd permutation or an even permutation according as whether r is an odd number or an even number. The other assertion follows from the remark made after Definition 16.4. □

We describe the "parity" of a product.

16.7 Theorem: *Let $n \geqslant 2$. The product of two permutations in S_n has the "parity" given by the following law.*

$$(odd)(odd) = (even) \qquad (odd)(even) = (odd)$$
$$(even)(odd) = (odd) \qquad (even)(even) = (even).$$

Proof: Let $\sigma,\pi \in S_n$. We want to find the "parity" of $\sigma\pi$. Let $\sigma = \tau_1\tau_2 \dots \tau_s$ and $\pi = \tau_1'\tau_2' \dots \tau_p'$, where $\tau_1,\tau_2, \dots,\tau_s,\tau_1',\tau_2', \dots,\tau_p'$ are transpositions (Theorem 16.2). Then $\sigma\pi = \tau_1\tau_2 \dots \tau_s\tau_1'\tau_2' \dots \tau_p'$ is a product of $s + p$ transpositions.

If σ is an odd permutation and π is an odd permutation, then s is an odd number and p is an odd number (Theorem 16.6), so $s + p$ is an even number, so $\sigma\pi$ is an even permutation (Theorem 16.6). Thus (odd)(odd) = (even). The other cases are proved similarly. □

The assertion of Theorem 16.7 resembles the rule for finding the sign of a product of two real numbers: the product of a negative number by a negative number is positive, etc. In order to exploit this analogy, we introduce a new term.

16.8 Definition: Let $n \in \mathbb{N}$ and $\sigma \in S_n$. The *sign of* σ is the integer 1 or -1. We write $\mathbb{E}(\sigma)$ for the sign of σ, and define it as follows.

$$\mathbb{E}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

With this definition, the content of Theorem 16.7 can be expressed more succintly.

16.7 Theorem: *For any σ,π in S_n, there holds $\mathbb{E}(\sigma\pi) = \mathbb{E}(\sigma)\mathbb{E}(\pi)$.*

□

16.9 Theorem: *Let $n \geqslant 2$. The number of odd permutations in S_n is equal to the number of even permutations in S_n. This number is $n!/2$.*

Proof: We must find a one-to-one correspondence between the set of odd permutations and the set of even permutations in S_n. Now

$$T: \{\sigma \in S_n: \mathbb{E}(\sigma) = -1\} \to \{\sigma \in S_n: \mathbb{E}(\sigma) = 1\}$$
$$\sigma \quad \to \quad (12)\sigma$$

is a one-to-one mapping (by Lemma 8.1(1)) from the set of odd permutations in S_n into the set of even permutations in S_n (by Lemma 16.3), which is in fact onto, since any even permutation π is the image, under T, of the odd permutation $(12)\pi$ (Lemma 16.3). So T is a one-to-one corre-spondence between these sets and they contain equal number of ele-ments, say k elements. Since these sets are disjoint, and their union is S_n, there are $2k$ elements in S_n, whose order is $n!$ by Theorem 15.2. Hence $k = n!/2$.

□

Theorem 16.7 asserts that the set of even permutations in S_n is closed under multiplication. So it is a subgroup of S_n by Lemma 9.3(2).

16.10 Definition: The subgroup of even permutations in S_n ($n \geqslant 2$) is called the *alternating group* (*on n letters*) and is written as A_n.

16.11 Theorem: *For $n \geqslant 2$, A_n is a group of order $n!/2$.*

Proof: Theorem 16.9. □

Exercises

1. Find the sign of (13524) and of (153462).

2. Show that a cycle of length m is odd (even) if and only if m is even (odd).

3. Prove that $\mathbb{E}(\sigma_1\sigma_2\ldots\sigma_t) = \mathbb{E}(\sigma_1)\mathbb{E}(\sigma_2)\ldots\mathbb{E}(\sigma_t)$ for all permutations $\sigma_1, \sigma_2, \ldots \sigma_t$ in S_n.

4. Find the sign of (143)(1245)(243) and of (1435)(25643) without evaluating these products.

5. Write all elements in A_2, A_3, A_4.

6. Construct multiplication tables of A_2, A_3, A_4.

7. Find all subgroups of A_4. Does A_4 have a subgroup of order 6?

8. Verify Lemma 16.3 by going through the argument in its proof in the specific cases below.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}, \ (ik) = (12), (14), (23), (26), (27), (67).$$