§18
Factor Groups

In this paragraph, we learn a way of constructing new groups from a given one. This construction is a generalization of obtaining the additive group $\mathbb{Z}_n$ from the additive group $\mathbb{Z}$. We recall that the elements of $\mathbb{Z}_n$ are certain subsets of $\mathbb{Z}$, namely the cosets of the subgroup $\{nz: z \in \mathbb{Z}\}$ in $\mathbb{Z}$ (cf. §10, Ex. 3). Addition in $\mathbb{Z}_n$ is induced from addition in $\mathbb{Z}$ (see §6). We want to do the same thing with an arbitrary group G. We start with a group G and a subgroup H of G. On the set of cosets of H in G, we wish to define a binary operation which reflects the operation on G and which makes the set of cosets into a group.

Two questions present themselves immediately. First, we have a set $\mathfrak{R}$ of right cosets of H in G and a set $\mathfrak{L}$ of left cosets of H in G. If G is an abelian group, the right cosets and the left cosets coincide. However, in general, the right cosets of H in G are different from the left cosets of H in G. Thus we have two different sets of cosets: $\mathfrak{R} \neq \mathfrak{L}$. Do we want to make $\mathfrak{R}$ into a group or $\mathfrak{L}$? Is it possible to make both $\mathfrak{R}$ and $\mathfrak{L}$ into groups? If so, how are these groups related? If not, why not?

Another question is about the operation. The central issue in §6, where we introduced the operations on $\mathbb{Z}_n$, was whether these operations were well defined. Once we knew that addition in $\mathbb{Z}_n$ is a well defined operation, it was straightforward to prove that $\mathbb{Z}_n$ is a group. Not surprisingly, we have the same problem here. The main point of the following discussion is to show that we have a well defined operation (Theorem 18.4). Once we know it, it is easy to show that our set of cosets is a group (Theorem 18.7).

It turns out that these questions are intimately connected and they will be resolved simultaneously.

18.1 Suggestion: *Let G be a group, H a subgroup of G, and let $\mathfrak{R}$ be the set $\{Ha: a \in G\}$ of all right cosets of H in G. We suggest that we define a binary operation on $\mathfrak{R}$, to be denoted by . or by juxtaposition, according to the "rule"*

$$Ha.Hb = Hab$$

for all $a,b \in G$.

This is the most natural way of defining a binary operation on $\Re$. Now we have to ask whether this is a well defined operation on $\Re$, for the "rule" of evalutating a product $Ha.Hb$ makes use of the elements a,b of G, which can be chosen in many ways. The "rule" says, in order to evaluate the product $X.Y$ of X and Y in $\Re$, that we (1) take an $a \in X$ so that $X = Ha$; (2) that we take an $b \in Y$ so that $Y = Hb$; (3) that we evaluate ab in G; (4) that we find the right coset $Hab \in \Re$ of this ab. The right coset Hab is supposed to be the product $X.Y$. We must make sure that we get the same right coset at the end, even if we choose different elements from the right cosets X and Y. We investigate when this "rule" yields a well defined operation on $\Re$.

The operation suggested in 18.1 is well defined if and only if the implication

$$\text{for all } a,a_1,b,b_1 \in G, \qquad Ha = Ha_1 \text{ and } Hb = Hb_1 \quad \Rightarrow \quad Hab = Ha_1b_1$$

is valid. Using Lemma 10.2, we write this in the equivalent form

$$\text{for all } a,a_1,b,b_1 \in G,\ h,h_1 \in H, \quad a_1 = ha \ \text{ and } \ b_1 = h_1b \quad \Rightarrow \quad a_1b_1 \in Hab$$

which simplifies to

$$\text{for all } a,a_1,b,b_1 \in G,\ h,h_1 \in H, \qquad hah_1b \in Hab.$$

Using Lemma 10.2 again, we can write this as

$$\text{for all } a \in G,\ h_1 \in H, \quad ah_1 \in Ha$$

or as

$$\text{for all } a \in G, \qquad aH \subseteq Ha. \tag{o}$$

Thus the operation sugested in 18.1 is well defined if and only if H is a subgroup of G such that $aH \subseteq Ha$ for all $a \in G$. This is not true for every G and for every subgroup H of G. After we give other descriptions of such subgroups, we will see some examples.

18.2 Lemma: *Let $H \leqslant G$. For $a \in G$, let $a^{-1}Ha$ be the set*

$$\{a^{-1}ha \in G: h \in H\} = \{b \in G: aba^{-1} \in H\}.$$

The following are equivalent.

(1) $a^{-1}ha \in H$ *for all* $a \in G, h \in H$.
(2) $a^{-1}Ha \subseteq H$ *for all* $a \in G$.
(3) $a^{-1}Ha = H$ *for all* $a \in G$.
(4) $Ha = aH$ *for all* $a \in G$.
(5) $aH \subseteq Ha$ *for all* $a \in G$.

Proof: (1) $\Rightarrow$ (2) This follows from the definition of the set $a^{-1}Ha$.

(2) $\Rightarrow$ (3) Suppose $a^{-1}Ha \subseteq H$ for all $a \in G$. Then, for any $a \in G$, it is true that $(a^{-1})^{-1}Ha \subseteq H$. Hence, for any $h \in H$, $a \in G$, we have $aha^{-1} \in H$, so $h = a^{-1}(aha^{-1})a \in a^{-1}Ha$. Since this holds for all $h \in H$, we obtain $H \subseteq a^{-1}Ha$, for all $a \in G$. Together with the hypothesis $a^{-1}Ha \subseteq H$ for all $a \in G$, this yields $a^{-1}Ha = H$ for all $a \in G$.

(3) $\Rightarrow$ (4) If $a^{-1}Ha = H$, then

$$\begin{aligned} Ha &= \{ha \in G: h \in H\} \\ &= \{a(a^{-1}ha) \in G: h \in H\} \\ &= \{ax \in G: x \in a^{-1}Ha\} \\ &= \{ax \in G: x \in H\} \\ &= aH. \end{aligned}$$

(4) $\Rightarrow$ (5) This is trivial.

(5) $\Rightarrow$ (1) Suppose $aH \subseteq Ha$ for all $a \in G$. Then $a^{-1}H \subseteq Ha^{-1}$ for all $a \in G$. Keeping a fixed, we see $a^{-1}h \in Ha^{-1}$ for all $h \in H$. Thus, for all $h \in H$, there is an $h_1 \in H$ such that $a^{-1}h = h_1a^{-1}$. So $a^{-1}ha = h_1 \in H$. So $a^{-1}ha \in H$ for all h in H, and this holds for all $a \in G$. □

18.3 Definition: Let $H \leqslant G$. If H satisfies one (and hence all) of the conditions in Lemma 18.2, then H is called a *normal subgroup of G*, or *normal in G*.

We employ the symbol $H \trianglelefteq G$ to denote that H is a normal subgroup of G. Also, $H \ntrianglelefteq G$ means that H is not a normal subgroup of G. If H is a proper and normal subgroup of G, we write $H \triangleleft G$. Finally, $H \ntriangleleft G$ means that H is not a proper normal subgroup of G.

18.4 Theorem: *The operation suggested in* 18.1 *is well defined if and only if* $H \trianglelefteq G$.

Proof: This follows from (o), Lemma 18.2(5) and Definition 18.3. □

By Lemma 18.2(4), any right coset of H in G is a left coset of H in G if and only if $H \trianglelefteq G$. So the set $\mathfrak{R}$ of right cosets of H is equal to the set $\mathfrak{L}$ of left cosets of H if and only if $H \trianglelefteq G$. Theorem 18.4 shows that we have a well defined operation on $\mathfrak{R}$ if and only if $\mathfrak{R} = \mathfrak{L}$. This answers our two questions. We do not have to bother about the distinction between $\mathfrak{R}$ and $\mathfrak{L}$: if (and only if) the operation is well defined, there is no distinction between $\mathfrak{R}$ and $\mathfrak{L}$. See also Ex. 1 at the end of this paragraph.

18.5 Examples: (a) For any group G, it is clear that $G \trianglelefteq G$. Also, $\{1\} \trianglelefteq G$, since $a^{-1}1a \in \{1\}$ for all $a \in G$. We make a convention here. The trivial subgroup $\{1\}$ will henceforward be written simply as 1. It will be clear from the context whether 1 stands for the identity element or for the trivial subgroup. Thus $1 \trianglelefteq G$ and $G \trianglelefteq G$.

(b) Any subgroup of an abelian group is normal in that group. Indeed, if G is abelian and $H \leqslant G$, then $hg = gh$ for all $h \in H$, $g \in G$, hence $Hg = gH$ for all $g \in G$. Thus $H \trianglelefteq G$ by Lemma 18.2(4).

In the abelian group case, $Hg = gH$ is satisfied trivially, for $hg = gh$ for all $h \in H$, $g \in G$. You should notice, however, $Hg = gH$ does *not* mean that g commutes with every element of H. This is an equation between certain sets, so is equivalent to the inclusions $Hg \subseteq gH$ and $gH \subseteq Hg$. The first inclusion means

$$\text{for all } h \in H, \text{ there is } h_1 \in H \text{ such that } hg = gh_1.$$

Here $h_1 \neq h$ in general and therefore $hg = gh_1 \neq gh$. The second inclusion has a similar meaning.

$Hg = gH$ means that, when we multiply the elements of H by g on the right and on the left, we get the same *collection* of elements. It does not mean that, when we multiply any element of H by g on the right and on the left, we get the same product.

Many beginners misunderstand this point. Be careful not to read more than set equality in $Hg = gH$. Compare this with an isometry fixing a subset F of the Euclidean plane E and one fixing F pointwise (§14).

(c) Consider the subgroup $A_3 = \{\iota,(123),(132)\}$ of S_3. There are $|S_3{:}A_3| = |S_3|/|A_3| = 6/3 = 2$ right cosets and 2 left cosets of A_3 in S_3. These are

$$A_3 \text{ and } A_3(12) = \{(12),(23),(13)\}$$
$$A_3 \text{ and } (12)A_3 = \{(12),(13),(23)\}$$

and so any right coset of A_3 in S_3 is also a left coset of A_3 in S_3. Thus $A_3 \trianglelefteq S_3$.

(d) The result in Example 18.5(c) can be generalized. Let $H \leqslant G$ of index $|G{:}H| = 2$. Then there are two right cosets of H in G and two left cosets of H in G. Let H and X be the right cosets, H and Y the left cosets. From the disjoint unions

$$G = H \cup X \quad \text{and} \quad G = H \cup Y,$$

we read off

$$X = G \setminus H = Y,$$

so the right cosets H,X of H in G coincide with the left cosets H,X of H in G. Hence $H \trianglelefteq G$: if H has index two in G, then H is normal in G.

(e) Consider the subgroup $H := \{\iota,(12)\}$ of S_3. Now $|S_3{:}H| = 6/2 = 3$. The three right cosets of H and the three left cosets of H are

$$\begin{array}{rlcrl} H &= \{\iota,(12)\} & \qquad & H &= \{\iota,(12)\} \\ H(13) &= \{(13),(123)\} & & (13)H &= \{(13),(132)\} \\ H(23) &= \{(23),(132)\} & & (23)H &= \{(23),(123)\} \end{array}$$

and the right coset $\{(13),(123)\}$ is not a left coset. So $H \ntrianglelefteq S_3$. In the same way, $\{\iota,(13)\}$ and $\{\iota,(23)\}$ are not normal subgroups of S_3.

(f) Let $H = \{\iota,(12),(34),(12)(34)\}$. It is easy to see that $H \leqslant S_4$. Is H normal in S_4? We compare the right and left cosets of H in S_4. Aside from H, we see that the right coset

$$H(13)(24) = \{(13)(24),(1423),(3241),(14)(23)\}$$

is a left coset:

$$(13)(24)H = \{(13)(24),(1324),(1423),(14)(23)\}$$

since $(3241) = (1324)$. This is of course not enough to conclude $H \trianglelefteq S_4$. We must examine the other cosets also. We see

$$H(13) = \{(13),(123),(341),(1234)\}$$
$$(13)H = \{(13),(132)$$

and we stop here. This shows $H(13) \neq (13)H$. Hence $H \ntrianglelefteq S_4$.

(g) Let $V_4 = \{\iota,(12)(34),(13)(24),(14)(23)\}$. It is easily seen that $V_4 \leqslant S_4$. The subgroup V_4 is known as *Klein's four group* (after the German mathematician Felix Klein (1849-1925); Vierergruppe, whence V_4). The cosets of V_4 in S_4 are

V_4 $\quad$ V_4
$V_4(12) = \{(12),(34),(1324),(1423)\}$, $(12)V_4 =$
$\{(12),(34),(1423),(1324)\}$
$V_4(13) = \{(13),(1234),(24),(1432)\}$, $(13)V_4 =$
$\{(13),(1432),(24),(1234)\}$
$V_4(23) = \{(23),(1342),(1243),(14)\}$, $(23)V_4$
$=\{(23),(2431),(2134),(14)\}$
$V_4(123) =\{(123),(134),(243),(142)\}$, $(123)V_4 =$
$\{(123),(243),(142),(134)\}$
$V_4(132) =\{(132),(234),(124),(143)\}$, $(132)V_4 =$
$\{(132),(143),(234),(124)\}$

and since each right coset is a left coset, $V_4 \trianglelefteq S_4$. For a more conceptual proof of this result, see Ex. 5 at the end of this paragraph.

(h) Consider $K = \{\iota,(12),(13),(23),(123),(132)\} \leqslant S_4$. Is K normal in S_4? We observe $(14)^{-1}K(14) = (14)K(14) = \{\iota,(42),(43),(23),(423),(432)\} \neq K$ and so $K \ntrianglelefteq S_4$.

(i) Normality is not an intrinsic property of a subgroup. It is meaningles to speak about normality of a subgroup H itself. It is only meaningful to speak about normality of H in a group G. We have to specify the group G

as well as the subgroup H when we speak about normality. It is possible that $H \trianglelefteq G_1$ and $H \ntrianglelefteq G_2$ for two groups G_1,G_2 containing H. Here is an example. Take

$$G_1 = D_8 = \langle \rho,\sigma : \rho^4 = 1, \sigma^2 = 1, \sigma^{-1}\rho\sigma = \rho^{-1}\rangle$$
$$G_2 = \langle \rho^2,\sigma\rangle = \{1,\rho^2,\sigma,\rho^2\sigma\} \leqslant G_1$$
$$H = \langle\sigma\rangle = \{1,\sigma\}.$$

Then $H \leqslant G_1$ and $H \leqslant G_2$. Now $|G_2{:}H| = 2$, so $H \trianglelefteq G_2$ by Example 18.5(d) above. However

$$\rho^{-1}H\rho = \{1,\rho^{-1}\sigma\rho\} = \{1,\rho^{-1}\rho^{-1}\sigma\} = \{1,\rho^2\sigma\} \neq H$$

and thus $H \ntrianglelefteq G_1$.

Incidentally, $G_2 \trianglelefteq G_1$ since $|G_1{:}G_2| = 2$. This shows that normality is not a transitive relation. It is possible that $H \trianglelefteq G_2$, $G_2 \trianglelefteq G_1$, yet $H \ntrianglelefteq G_1$.

(j) For any field K, we have $SL(2,K) \trianglelefteq GL(2,K)$. Indeed, if $S \in SL(2,K)$, then $\det S = 1$ and, for any $G \in GL(2,K)$,

$$\begin{aligned} \det(G^{-1}SG) &= \det(G^{-1}.SG) = \det G^{-1}.\det(SG) = (\det G)^{-1}.(\det S)(\det G) \\ &= (\det G)^{-1}1(\det G) = 1, \end{aligned}$$
$$G^{-1}SG \in SL(2,K) \text{ for all } S \in SL(2,K),\ G \in GL(2,K),$$

and so $SL(2,K) \trianglelefteq GL(2,K)$ by Lemma 18.2(1).

(k) If $H \trianglelefteq G$ and $K \trianglelefteq G$, then $H \cap K \trianglelefteq G$. More generally, if $H_i \trianglelefteq G$ (where $i \in I$, an index set), then $\bigcap_{i\in I} H_i \trianglelefteq G$. We show this. Put $H = \bigcap_{i\in I} H_i$ for brevity. From $H_i \leqslant G$, it follows that $H \leqslant G$ (Example 9.4(f)). Also, for any $h \in H$ and $g \in G$,

$$\begin{aligned} h &\in H_i \text{ for all } i \in I, \\ g^{-1}hg &\in H_i \text{ for all } i \in I, \\ g^{-1}hg &\in H \end{aligned}$$

and $H \trianglelefteq G$ by Lemma 18.2(1).

(l) If $H \trianglelefteq G$ and $K \leqslant G$, then $H \cap K \trianglelefteq K$. Indeed, let $h \in H \cap K$, $k \in K$. Then $k^{-1}hk \in H$ since $h \in H$ and H is normal in G. Also, $k^{-1}hk \in K$ because $h \in K$ and K is closed under multiplication. Thus $k^{-1}hk \in H \cap K$ for all $h \in H \cap K$ and for all $k \in K$. Thus $H \cap K \trianglelefteq K$ by Lemma 18.2(1).

18.6 Definition: When $H \trianglelefteq G$, the set of all right cosets of H in G, which is also the set of all left cosets of H in G by Lemma 18.2(4), will be denoted by G/H, read *G by H*, or *G modulo H*, or *G mod H*.

Most authors do not insist on the condition $H \trianglelefteq G$ when they write G/H. They write G/H for the set $\mathfrak{R}$ of right cosets of H in G (or for the set of left cosets, especially when they write functions on the left) and employ some other symbol for the the set of left cosets (or for the the set of right cosets). Throughout this book, whenever we write G/H, it will be tacitly supposed that $H \trianglelefteq G$. The notation G/H is meaningless if $H \ntrianglelefteq G$ and will not be used in this case.

18.7 Theorem: *Let $H \trianglelefteq G$. Then G/H is a group under the operation suggested in* 18.1, *by which*

$$Ha.Hb = Hab \quad \text{for all } Ha, Hb \in G/H.$$

Proof: We check the group axioms.

(i) The operation on G/H is well defined by Theorem 18.4 and the product of two right cosets is again a right coset. So G/H is closed under this operation.

(ii) For all $Ha,Hb,Hc \in G/H$, we have $(Ha.Hb)Hc = Hab.Hc = H(ab.c) = H(a.bc) = Ha.Hbc = Ha(Hb.Hc)$ since $ab.c = a.bc$ for all $a,b,c \in G$. The operation is therefore associative.

(iii) $H = H1 \in G/H$ is a right identity element of since

$$Ha.H1 = Ha1 = Ha \text{ for all } Ha \in G/H.$$

(iv) Any $Ha \in G/H$ has a right inverse in G/H, namely Ha^{-1}:
$Ha.Ha^{-1} = Ha.a^{-1} = H1 = H =$ identity element of G/H.

Therefore G/H is a group. □

18.8 Definition: Let $H \trianglelefteq G$. The group G/H of Theorem 18.7 is called the *factor group of G with respect to H*, or the *factor group G by H*, or the *factor group G mod*(*ulo*) *H*. Instead of the term "factor group", the term "quotient group" is also used. The group operation is called *multiplication* (*of cosets*).

Please notice that G/H is *not* a subgroup of G. The elements of G/H are subsets of G, not elements of G.

Since the multiplication on G/H is based on the multiplication on G, we expect that some properties of G are inherited by G/H. Here are some properties that are taken over by the factor groups.

18.9 Lemma: *Let* $H \trianglelefteq G$.
(1) $|G/H| = |G{:}H|$. *In particular, if* G *is finite, so is* G/H *and* $|G/H| = |G|/|H|$.
(2) *If* G *is abelian, so is* G/H.
(3) *If* G *is cyclic, so is* G/H.

Proof: (1) The elements of G/H are the cosets of H in G and there are $|G{:}H|$ cosets of H in G by Definition 10.7. So the order of G/H is the index of H in G. The second assertion follows from Lagrange's theorem.

(2) If G is abelian, then $ab = ba$ for all $a,b \in G$ and $Ha.Hb = Hab = Hba = Hb.Ha$ for all $Ha, Hb \in G/H$. Thus G/H is abelian, too.

(3) Assume that G is cyclic, say $G = \langle g \rangle$. Then any element x of G is of the form g^n, where $n \in \mathbb{Z}$. Hence any coset of H in G is of the form $Hx = Hg^n = (Hg)^n$. This shows $G/H = \langle Hg \rangle$. □

The converses of the claims in Lemma 18.9 are false. The factor group G/H can be finite (abelian, cyclic) without G being finite (abelian, cyclic).

We close this paragraph with some examples of factor groups.

18.10 Examples: (a) Let G be a group and $H = 1 = \{1\}$. Then $H \trianglelefteq G$ (Example 18.5(a)). The cosets of $H = 1$ the subsets of G having only one element:

$$Ha = \{1\}a = \{a\} \quad \text{for all } a \in G$$

and multiplication in $G/H = G/1$ is given by

$$\{a\}\{b\} = \{ab\}.$$

The factor group $G/1$ is governed by the same operation as G. Thus $G/1$ is almost the same group as G. The only difference is that the elements of G are enclosed within braces in $G/1$.

(b) Let $\mathbb{Z}$ be the additive group of integers and let $n\mathbb{Z} = \{nz \in \mathbb{Z} : z \in \mathbb{Z}\}$ be the subgroup of $\mathbb{Z}$ consisting of integers divisible by n. Since $\mathbb{Z}$ is abelian, $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ consists of the *n cosets*

$$n\mathbb{Z}, n\mathbb{Z} + 1, n\mathbb{Z} + 2, \ldots, n\mathbb{Z} + n - 1$$

which are usually abbreviated as

$$\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}$$

(see §6; we write the cosets additively of course). Thus $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ as sets.

In the factor group $\mathbb{Z}/n\mathbb{Z}$, the operation is given by

$$(n\mathbb{Z} + a) + (n\mathbb{Z} + b) = n\mathbb{Z} + (a + b) \qquad \text{for all } a,b \in \mathbb{Z}$$

which can be written shortly as

$$\overline{a} + \overline{b} = \overline{a+b} \qquad \text{for all } \overline{a},\overline{b} \in \mathbb{Z}/n\mathbb{Z}.$$

This is the definition of addition in $\mathbb{Z}_n$. So the operation in $\mathbb{Z}/n\mathbb{Z}$ coincides with the operation on $\mathbb{Z}_n$ that we learned in §6. Hence $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ as (additive) groups.

We understand the real reason why addition on $\mathbb{Z}_n$, as defined in §6, is a well defined operation. It is well defined only because $n\mathbb{Z} \trianglelefteq \mathbb{Z}$.

(c) Let $G = C_{12} = \langle g: g^{12} = 1\rangle$ be a cyclic group of order 12 and let $H = \langle g^3\rangle = \{1,g^3,g^6,g^9\} \leqslant G$. Since G is abelian, $H \trianglelefteq G$ and G/H consists of the cosets

$$H = \{1,g^3,g^6,g^9\},\ Hg = \{g,g^4,g^7,g^{10}\},\ Hg^2 = \{g^2,g^5,g^8,g^{11}\}.$$

The multiplication table of G/H is given below.

	H	Hg	Hg^2
H	H	Hg	Hg^2
Hg	Hg	Hg^2	H
Hg^2	Hg^2	H	Hg

(d) We know $V_4 \trianglelefteq S_4$ (Example 18.5(g)). The elements of S_4/V_4 are $V_4(12), V_4(13), V_4(23), V_4(123), V_4(132)$ and the multiplication table of S_4/V_4 is

.	V_4	$V_4(12)$	$V_4(13)$	$V_4(23)$	$V_4(123)$	$V_4(132)$
V_4	V_4	$V_4(12)$	$V_4(13)$	$V_4(23)$	$V_4(123)$	$V_4(132)$
$V_4(12)$	$V_4(12)$	V_4	$V_4(123)$	$V_4(132)$	$V_4(13)$	$V_4(23)$
$V_4(13)$	$V_4(13)$	$V_4(132)$	V_4	$V_4(123)$	$V_4(23)$	$V_4(12)$
$V_4(23)$	$V_4(23)$	$V_4(123)$	$V_4(132)$	V_4	$V_4(12)$	$V_4(13)$
$V_4(123)$	$V_4(123)$	$V_4(23)$	$V_4(12)$	$V_4(13)$	$V_4(132)$	V_4
$V_4(132)$	$V_4(132)$	$V_4(13)$	$V_4(23)$	$V_4(12)$	V_4	$V_4(123)$

This is almost identical with with the multiplication table of S_3:

	ι	(12)	(13)	(23)	(123)	(132)
ι	ι	(12)	(13)	(23)	(123)	(132)
(12)	(12)	ι	(123)	(132)	(13)	(23)
(13)	(13)	(132)	ι	(123)	(23)	(12)

(23)	(23)	(123)	(132)	ι	(12)	(13)
(123)	(123)	(23)	(12)	(13)	(132)	ι
(132)	(132)	(13)	(23)	(12)	ι	(123)

Thus S_4/V_4 is almost the same group as S_3. They are not the same groups, of course, for the underlying sets are different. Nevertheless, it is clear from the tables above that the operations on S_4/V_4 and on S_3 are closely related. This will be made more precise in §20.

Exercises

1. Let $H \leqslant G$ and let $\mathcal{L}$ be the set of all left cosets of H in G. We suggest that we define a binary operation on $\mathcal{L}$, according to the "rule"

$$aH.bH = abH$$

for all $a,b \in G$. Show that this operation is well defined if and only if $H \trianglelefteq G$.

2.Let $H \leqslant G$. Prove that $H \trianglelefteq G$ if and only if $Ha \subseteq aH$ for all $a \in G$.

3. Prove that, if $H \leqslant G$, $a \in G$ and Ha is a left coset of H in G, then $Ha = aH$.

4. Find a group G, a subgroup H of G, and an element a of G such that $a^{-1}Ha \subseteq H$ but $a^{-1}Ha \neq H$. Why does this not contradict Lemma 18.2?

5. Let $\{a,b,c,d\} = \{1,2,3,4\}$. Show that, for any $\sigma \in S_4$,

$$(ab)(cd)\sigma = \sigma(a\sigma,b\sigma)(c\sigma,d\sigma)$$

and thus $\sigma^{-1}\alpha\sigma \in V_4$ for all $\alpha \in V_4$. This proves $V_4 \trianglelefteq S_4$. Compare with §15, Ex. 15.

6. Find all normal subgroups of S_4 (cf. §15, Ex.10).

7. Find all normal subgroups of $SL(2,\mathbb{Z}_3)$.

8. Determine whether the following are normal subgroups in the groups indicated.

$\{g \in GL(2,\mathbb{R})\colon det\ g \geqslant 5\}$ in $GL(2,\mathbb{R})$
$\{g \in GL(2,\mathbb{R})\colon det\ g \geqslant 0\}$ in $GL(2,\mathbb{R})$
$\{g \in GL(2,\mathbb{R})\colon det\ g > 0\}$ in $GL(2,\mathbb{R})$
$\{g \in GL(2,\mathbb{C})\colon det\ g = 1\}$ in $GL(2,\mathbb{C})$
$\{g \in GL(2,\mathbb{C})\colon (det\ g)^{18} = 1\}$ in $GL(2,\mathbb{C})$
$\{g \in GL(2,\mathbb{Z}_{11})\colon det\ g = \overline{1}$ or $\overline{3}$ or $\overline{4}$ or $\overline{5}$ or $\overline{9}\}$ in $GL(2,\mathbb{Z}_{11})$.

9. Let K be a field. Then $K\backslash\{0\}$ is a group under multiplication. Suppose U is a subgroup of $K\backslash\{0\}$. Prove that $\{g \in GL(2,K)\colon det\ g \in U\}$ is a subgroup of $GL(2,K)$.

10. Let $n \in \mathbb{N}$ and put

$$\Gamma_n = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\colon \begin{matrix} a \equiv 1 & b \equiv 0 \\ c \equiv 0 & d \equiv 1 \end{matrix} \pmod{n}\right\}.$$

Determine if $\Gamma_n \trianglelefteq SL(2,\mathbb{Z})$.

11. Let $H \trianglelefteq G$ and let $Ha \in G/H$. Show that $o(Ha) = n$ (n is a natural number) if and only if n is the smallest natural number such that $x^n \in H$.

12. Show by counterexamples that the converses of the claims in Lemma 18.9 are false.