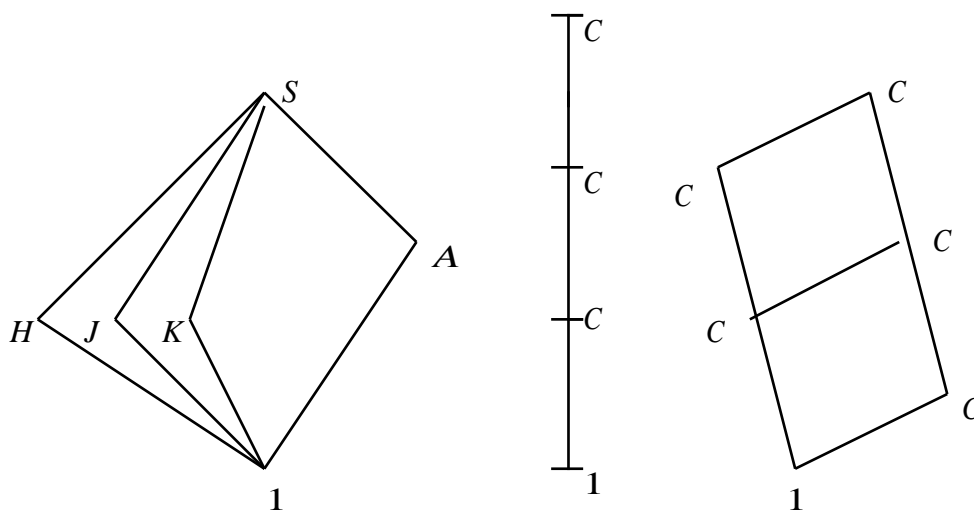


## §21 Isomorphism Theorems

This paragraph is devoted to some very important theorems of group theory.

At this stage, it will be useful to introduce Hasse diagrams. A Hasse diagram is a convenient means to visualise inclusions holding between subgroups of a group. Subgroups are represented by points. Two points (subgroups) are joined by a line segment if and only if the lower subgroup is contained in the upper one. The line segments may be vertical or slant. The line segments may be thought of as factor groups when the lower subgroup is normal in the upper one. The Hasse diagrams of  $S_3$ ,  $C_8$  and  $C_{12}$  are depicted below.



$$H = \{1, (12)\}, J = \{1, (13)\}, K = \{1, (23)\}.$$

**21.1 Theorem:** *Let  $\varphi: G \rightarrow G_1$  be a group homomorphism from  $G$  onto  $G_1$ . Then there is a one-to-one correspondence between the set of all subgroups of  $G$  that contain  $\text{Ker } \varphi$  and the set of all subgroups of  $G_1$ . This correspondence preserves inclusion. The normal subgroups of  $G$  that contain  $\text{Ker } \varphi$  correspond to normal subgroups of  $G_1$ , and conversely. The factor groups by corresponding normal subgroups are isomorphic.*

In more detail and more precise language, the claim is the following.

- (1) For every  $H \leq G$  with  $\text{Ker } \varphi \leq H$ , there is associated a unique subgroup of  $G_1$ , which will be denoted by  $H_1$ .
- (2) If  $\text{Ker } \varphi \leq H \leq J \leq G$ , then  $H_1 \leq J_1$ .
- (3) If  $\text{Ker } \varphi \leq H \leq G$ ,  $\text{Ker } \varphi \leq J \leq G$  and  $H_1 \leq J_1$ , then  $H \leq J$ .
- (4) If  $\text{Ker } \varphi \leq H \leq G$ ,  $\text{Ker } \varphi \leq J \leq G$  and  $H_1 = J_1$ , then  $H = J$ .
- (5) If  $S$  is any subgroup of  $G_1$ , then there is an  $H \leq G$  such that  $\text{Ker } \varphi \leq H$  and  $H_1 = S$ .
- (6) For  $\text{Ker } \varphi \leq H \leq G$ , there holds  $H \trianglelefteq G$  if and only if  $H_1 \trianglelefteq G_1$ .
- (7) If  $\text{Ker } \varphi \leq H \trianglelefteq G$  and  $H_1 \trianglelefteq G_1$ , then  $G/H \cong G_1/H_1$ .

The situation is described in the accompanying diagrams.

$$\begin{array}{c|c}
 G & G_1 \\
 \hline
 J & J_1 \\
 \hline
 H & H_1 \\
 \hline
 \text{Ker } \varphi & 1 \\
 \hline
 1 & 
 \end{array}
 \qquad
 \begin{array}{c|c|c}
 G & G_1 & G_1/H_1 \\
 \hline
 & & \\
 \hline
 H & H_1 & 1 \\
 \hline
 \text{Ker } \varphi & 1 & \\
 \hline
 1 & & 
 \end{array}$$

**Proof:** (1) For each  $H \leq G$  with  $\text{Ker } \varphi \leq H$ , we are to find a subgroup of  $G_1$ . How can we find it? Well, the subgroup we are looking for will be first of all a *subset* of  $G_1$ . How can we associate with  $H$  a subset of  $G_1$ ? At our disposal, we have only one means of transportation from  $G$  to  $G_1$ , namely the mapping  $\varphi$ . The only thing we can do, then, is form the set of images of the elements of  $H$  under  $\varphi$ . Hence we put

$$H_1 := \{h\varphi \in G_1 : h \in H\}.$$

We now prove  $H_1 \leq G_1$ . We can do it by the subgroup criterion, but we prefer to use Theorem 20.6, which states that the image of a homomorphism is a subgroup of its range. We note that the restriction of  $\varphi$  to  $H$  is a homomorphism (Example 20.2(g)) and

$$H_1 = \{h\varphi \in G_1 : h \in H\} = \{h\varphi_H \in G_1 : h \in H\} = \text{Im } \varphi_H$$

by definition. Theorem 20.6 gives now  $Im \varphi_H \leq G_1$ , hence  $H_1 \leq G_1$ . The description  $H_1 = Im \varphi_H$  will be useful.

(2) Suppose  $Ker \varphi \leq H \leq J \leq G$ . Under this assumption, we prove  $H_1 \leq J_1$ . This is easy: for any  $h \in H$ , we have  $h \in J$ , so  $h\varphi \in Im \varphi_J = J_1$ . Since  $h\varphi \in J_1$  for all  $h\varphi \in H_1$ , we get  $H_1 \leq J_1$ .

(3) Suppose  $Ker \varphi \leq H \leq G$ ,  $Ker \varphi \leq J \leq G$  and  $H_1 \leq J_1$ . We want to prove  $H \leq J$ . Now  $H_1 \leq J_1$  means  $Im \varphi_H \leq Im \varphi_J$ . Then, for every  $h \in H$ , we have  $h\varphi \in Im \varphi_J$ :

for every  $h \in H$ , there is a  $j \in J$  such that  $h\varphi = j\varphi$ .

We obtain, when  $h, j$  are as above

$$\begin{aligned} 1 &= h\varphi(j\varphi)^{-1} = h\varphi.j^{-1}\varphi = (hj^{-1})\varphi, \\ &hj^{-1} \in Ker \varphi \leq J \\ &h \in Jj = J. \end{aligned}$$

Thus  $h \in J$  for all  $h \in H$ . Therefore  $H \leq J$ .

(4) This is immediate from (3). If  $H_1 = J_1$ , we have  $H_1 \leq J_1$  and  $J_1 \leq H_1$ , so  $H \leq J$  and  $J \leq H$  by (3), hence  $H = J$ . (This shows that the correspondence  $H \rightarrow H_1$  is one-to-one.)

(5) For any  $S \leq G_1$ , we are to find an  $H \leq G$  such that  $Ker \varphi \leq H$  and  $H_1 = S$ . What can  $H$  be? As in part (1), there is only one thing we can do: take the preimages of the elements in  $S$ . Hence we put

$$H = \{a \in G: a\varphi \in S\}.$$

Thus  $a \in H$  means  $a\varphi \in S$ . We show that  $H \leq G$ , that  $Ker \varphi \leq H$  and that  $H_1 = S$ .

First  $H \leq G$ . From  $1_G\varphi = 1_{G_1} \in S$  (Lemma 20.3(1)), we get  $1 \in H$ . So  $H \neq \emptyset$ .

We apply the subgroup criterion.

(i) If  $a, b \in H$ , then  $a\varphi, b\varphi \in S$ , so  $a\varphi b\varphi \in S$ , so  $(ab)\varphi \in S$ , so  $ab \in H$ . Thus  $H$  is closed under multiplication.

(ii) If  $a \in H$ , then  $a\varphi \in S$ , then  $(a\varphi)^{-1} \in S$ , then  $(a^{-1})\varphi \in S$ , then  $a^{-1} \in H$ . Thus  $H$  is closed under the forming of inverses.

Thus  $H$  is a subgroup of  $G$ .

We prove next that  $H$  contains  $\text{Ker } \varphi$ . This is trivial. If  $a \in \text{Ker } \varphi$ , then  $a\varphi = 1$ , so  $a\varphi \in S$ , so  $a \in H$ . Hence  $\text{Ker } \varphi \leq H$ .

It remains to prove  $H_1 = S$ . We have

$$H_1 = \text{Im } \varphi_H = \{h\varphi \in G_1 : h \in H\} = \{h\varphi \in G_1 : h\varphi \in S\} = S$$

as claimed.

This completes the proof of (5). (Part (5) shows that the correspondence  $H \rightarrow H_1$  is onto.)

(6) First we assume  $H \triangleleft G$  and show that  $H_1 \triangleleft G_1$ . We are to show that  $x^{-1}h_1x \in H_1$  for all  $x \in G_1$  and for all  $h_1 \in H_1$  (Lemma 18.2(1)). If  $x \in G_1$  and  $h_1 \in H_1$ , then there are  $a \in G$  with  $a\varphi = x$  and  $h \in H$  with  $h\varphi = h_1$ . This is so because  $\varphi$  is onto  $G_1$  and  $H_1$  is defined as  $\text{Im } \varphi_H$ . Then we are to show  $(a\varphi)^{-1}(h\varphi)(a\varphi) \in H_1$ . This is equivalent to  $(a^{-1}ha)\varphi \in H_1$ . Since  $H \triangleleft G$ , we know  $a^{-1}ha \in H$ , so  $(a^{-1}ha)\varphi \in \text{Im } \varphi_H = H_1$ . This proves  $H_1 \triangleleft G_1$ .

We assume now  $H_1 \triangleleft G_1$  and prove  $H \triangleleft G$ . We can give an argument similar to the one above, but we prefer to use the fact that normal subgroups and kernels coincide. Our method will be used in the proof of part (7) as well.

The assumption is  $H_1 \triangleleft G_1$ . By Theorem 20.12,  $H_1 = \text{Ker } v'$ , where

$$v': G_1 \rightarrow G_1/H_1$$

is the natural homomorphism. We get then the homomorphism

$$\varphi v': G \rightarrow G_1/H_1 \quad (\text{Theorem 20.4):}$$

$$G \xrightarrow{\varphi} G_1 \xrightarrow{v'} G_1/H_1. \quad (\text{i})$$

We have

$$\begin{aligned} \text{Ker } \varphi v' &= \{a \in G : a(\varphi v') = H_1\} \\ &= \{a \in G : a\varphi \in \text{Ker } v'\} \\ &= \{a \in G : a\varphi \in H_1\} \end{aligned}$$

So  $(\text{Ker } \varphi v')_1 = \text{Im } \varphi_{\text{Ker } \varphi v'} = \{a\varphi \in G_1 : a \in \text{Ker } \varphi v'\} = \{a\varphi \in G_1 : a\varphi \in H_1\} = H_1$  and we obtain

$$\text{Ker } \varphi v' = H \quad (\text{ii})$$

by part (4). Theorem 20.6 gives  $H \triangleleft G$ , as was to be proved.

(7) We saw that any one of  $H \trianglelefteq G$  and  $H_1 \trianglelefteq G_1$  implies the other. Assume that one, and hence both of them are true. Then we have the homomorphism  $\varphi v'$ . From Theorem 20.16, we obtain

$$G/\text{Ker } \varphi v' \cong \text{Im } \varphi v' \quad (\text{iii})$$

We know  $\text{Ker } \varphi v' = H$  by (ii). As for the image, since  $\varphi$  is onto  $G_1$  by hypothesis and  $v'$  is onto  $G_1/H_1$  by Theorem 20.12, the composition  $\varphi v'$  is onto by Theorem 3.11(1). Hence  $\text{Im } \varphi v' = G_1/H_1$  and (iii) becomes

$$G/H \cong G_1/H_1.$$

The proof is complete. □

An important special case of Theorem 21.1 is the case of a natural homo-morphism, recorded in the next theorem. It gives a complete description of the subgroups of a factor group. The last part of the theorem is known as the factor of a factor theorem.

**21.2 Theorem:** *Let  $N \trianglelefteq G$ . The subgroups of  $G/N$  are the factor groups  $S/N$ , where  $S$  runs through the subgroups of  $G$  satisfying  $N \leq S$ . More precisely, for each subgroup  $X$  of  $G/N$ , there is a unique subgroup  $S$  of  $G$  satisfying  $N \leq S$  such that  $X = S/N$ . When  $X_1$  and  $X_2$  are subgroups of  $G/N$ , say  $X_1 = S_1/N$  and  $X_2 = S_2/N$ , where  $N \leq S_1 \leq G$  and  $N \leq S_2 \leq G$ , then  $X_1 \leq X_2$  if and only if  $S_1 \leq S_2$ . Furthermore,  $S/N \trianglelefteq G/N$  if and only if  $S \trianglelefteq G$ . In this case, there holds*

$$G/N / S/N \cong G/S.$$

**Proof:** Since  $N \trianglelefteq G$ , we can build the factor group  $G/N$ . The natural homomorphism  $v: G \rightarrow G/N$  is onto by Theorem 20.12. We can therefore apply Theorem 21.1.

Theorem 21.1 states that any subgroup of  $G/N$  is of the form  $\text{Im } v_S$  for some  $S \leq G$  with  $\text{Ker } v \leq S$  (here  $v_S$  is the restriction of  $v$  to  $S$ ). Now

$$\begin{aligned} \text{Im } v_S &= \{sv \in G/N: s \in S\} \\ &= \{Ns \in G/N: s \in S\} = S/N \end{aligned}$$

and  $\text{Ker } \nu = N$  by Theorem 20.12 (notice that  $S/N$  is meaningful, for  $N \trianglelefteq G$  and  $N \leq S$  imply  $N \trianglelefteq S$ ; cf. Example 18.5(1)). Thus the subgroups of  $G/N$  are given by  $S/N$ , where  $N \leq S \leq G$ . By Theorem 21.1(2),(3),(4),  $S_1/N \leq S_2/N$  if and only if  $S_1 \leq S_2$  and  $S_1/N \neq S_2/N$  whenever  $S_1 \neq S_2$ . Finally,  $S_1/N \trianglelefteq G/N$  if and only if  $S \trianglelefteq G$  by Theorem 21.1(6) and in this case  $G/N / S/N \cong G/S$  by Theorem 21.1(7). This completes the proof.  $\square$

$$\begin{array}{ccc|ccc} & G & & G/N & & G/N / S/N \\ & | & & | & & | \\ & S & & S/N & & 1 \\ & | & & | & & \\ & N & & 1 & & \\ & | & & & & \\ & 1 & & & & \end{array}$$

As an application of Theorem 21.2, we classify the factor groups of cyclic groups. We treat infinite and finite cyclic groups separately.

Any infinite cyclic group is isomorphic to  $\mathbb{Z}$  under addition (Example 20.17(a)), so we need find the factor groups of  $\mathbb{Z}$ . As  $\mathbb{Z}$  is abelian, any subgroup of  $\mathbb{Z}$  is normal in  $\mathbb{Z}$  (Example 18.5(b)) and we can build factor groups of  $\mathbb{Z}$  by any subgroup of  $\mathbb{Z}$ . The subgroups of  $\mathbb{Z}$  are  $0$  (see Example 18.5(a)) and  $n\mathbb{Z}$ , where  $n \in \mathbb{N}$  (Theorem 11.8). For each  $n \in \mathbb{N}$ , the subgroup  $n\mathbb{Z}$  is the unique subgroup of index  $n$  (Lemma 11.11). The factor group  $\mathbb{Z}/0$  is isomorphic to  $\mathbb{Z}$  (Example 20.10(c)). The factor groups  $\mathbb{Z}/n\mathbb{Z}$  are known to be cyclic of order  $n$  (Example 20.17(a)). So all factor groups of  $\mathbb{Z}$  are cyclic (cf. Lemma 18.9(3)). For each  $m \in \mathbb{N} \cup \{\infty\}$ , there is a unique factor group of order  $m$  of  $\mathbb{Z}$ , namely  $\mathbb{Z}/m\mathbb{Z}$  if  $m \in \mathbb{N}$  and  $\mathbb{Z}/0 \cong \mathbb{Z}$  if  $m = \infty$ .

Now let  $C_n = \langle a \rangle$  be a finite cyclic group of order  $n \in \mathbb{N}$ . As  $C_n$  is abelian, we can build factor groups of  $C_n$  by any subgroup of  $C_n$ . We know that  $C_n \cong \mathbb{Z}/n\mathbb{Z}$  from Example 20.17(a). The subgroups of  $\mathbb{Z}/n\mathbb{Z}$  are described in Theorem 21.2: any subgroup of  $\mathbb{Z}/n\mathbb{Z}$  is of the form  $M/n\mathbb{Z}$ , where  $n\mathbb{Z} \leq M \leq \mathbb{Z}$ . Now  $M \leq \mathbb{Z}$  means  $M = m\mathbb{Z}$  for some  $m \in \mathbb{N}$  or  $M = 0$  (Theorem 11.8) and the condition  $n\mathbb{Z} \leq M$  excludes  $M = 0$ . Hence  $M = m\mathbb{Z}$  for some  $m \in \mathbb{N}$ , where furthermore  $m|n$ , because  $n\mathbb{Z} \leq m\mathbb{Z}$ . So the sub-

groups of  $\mathbb{Z}/n\mathbb{Z}$  are given by  $m\mathbb{Z}/n\mathbb{Z}$ , where  $m$  runs through all positive divisors of  $n$ . For the factor group, we know

$$\mathbb{Z}/n\mathbb{Z}/m\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$$

from Theorem 21.2. So all factor groups of  $\mathbb{Z}/n\mathbb{Z}$  and of  $C_n$  are cyclic (cf. Lemma 18.9(3)). For each positive divisor  $m$  of  $n$ , there is a unique factor group of order  $m$  of  $C_n = \langle a \rangle$ , namely  $\langle a \rangle / \langle a^m \rangle$ , where  $\langle a^m \rangle$  is the unique subgroup of order  $n/m$  of  $C_n$  (Lemma 11.10).

$$\begin{array}{ccccccc}
 & \left| \begin{array}{c} C_n \\ \\ 1 \end{array} \right. & & \left| \begin{array}{c} \mathbb{Z}/n\mathbb{Z} \\ \\ 1 \end{array} \right. & & \left| \begin{array}{c} \mathbb{Z} \\ \\ n\mathbb{Z} \\ \\ 1 \end{array} \right. & & \left| \begin{array}{c} \mathbb{Z}/n\mathbb{Z} / m\mathbb{Z}/n\mathbb{Z} \\ \\ 1 \end{array} \right. \\
 \left. \langle a^m \rangle \right. & & & \left. m\mathbb{Z}/n \right. & \cong & & & \\
 & & & & & & & 
 \end{array}$$

We end this paragraph with another important theorem of group theory.

**21.3 Theorem:** Let  $H \triangleleft G$  and  $K \leq G$ . Then  $H \cap K \triangleleft K$  and

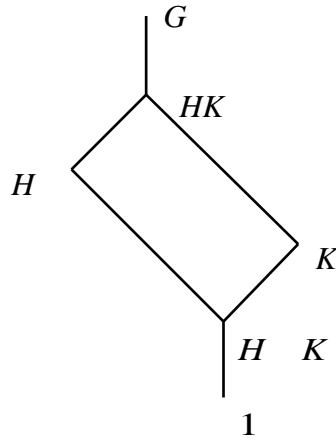
$$K / H \cap K \cong HK / H. \tag{*}$$

**Proof:** Since  $H \triangleleft G$ , there is a group  $G/H$  and a homomorphism

$$v: G \rightarrow G/H$$

Let  $v_K$  be the restriction of  $v$  to  $K$ . This  $v_K$  is a homomorphism (Example 20.2(g)). Hence  $K/\text{Ker } v_K \cong \text{Im } v_K$  by Theorem 20.16. Here

$$\begin{aligned}
 \text{Ker } v_K &= \{k \in K: kv = 1\} \\
 &= \{k \in K: k \in \text{Ker } v\} \\
 &= K \cap \text{Ker } v \\
 &= K \cap H.
 \end{aligned}$$



It remains to find  $\text{Im } \nu_K$ . We claim  $\text{Im } \nu_K = HK/H$ . First of all,  $HK = KH$  is a subgroup of  $G$  because  $H \triangleleft G$  (Lemma 19.4(2)), and  $H \leq HK$ , so  $H \triangleleft HK$ . So  $HK/H$  is meaningful. For any  $k \in K$ , we have  $k\nu_K = Hk \in HK/H$ , which shows that  $\text{Im } \nu_K \subseteq HK/H$ . Conversely, each element of  $HK/H$  is of the form  $Hhk$ , where  $h \in H, k \in K$ . But  $Hhk = Hk = k\nu_K \in \text{Im } \nu_K$ , so  $HK/H \subseteq \text{Im } \nu_K$ . Thus  $\text{Im } \nu_K = HK/H$  and (\*) yields

$$K / H \cap K \cong HK/H$$

as was to be proved. □

### Exercises

1. Let  $A \triangleleft C \leq G$  and  $B \leq G$ . Prove that  $A \cap B \triangleleft C \cap B$  and  $C \cap B / A \cap B \cong A(C \cap B)/A$ .
2. Let  $A \triangleleft C \leq G, B \leq G$  and let  $\varphi: G \rightarrow H$  be a group homomorphism. Prove that  $A\varphi \triangleleft C\varphi$ . Choosing  $\varphi$  in particular to be the natural homomorphism  $\nu: G \rightarrow G/B$ , prove that  $AB \triangleleft CB$ .