

§22 Direct Products

In this paragraph, we learn a method of constructing new groups from given ones. This method consists essentially in writing the groups one adjacent to the other.

22.1 Theorem: *Let H and K be groups. On the cartesian product $H \times K$, we define a binary operation by declaring*

$$(h,k)(h_1,k_1) = (hh_1,kk_1)$$

for all $(h,k),(h_1,k_1) \in H \times K$. With respect to this operation, $H \times K$ is a group.

Proof: Before beginning with the proof, it will not be amiss to formulate the theorem in a more precise way. Suppose (H, \circ) and $(K, *)$ are groups. The claim is that $(H \times K, \Delta)$ is a group, where Δ is defined by

$$(h,k) \Delta (h_1,k_1) = (h \circ h_1, k * k_1)$$

for all $(h,k),(h_1,k_1) \in H \times K$.

The multiplication in $H \times K$ is carried out componentwise. Since H and K are groups themselves, it is natural to expect that $H \times K$ will be a group. We check the group axioms.

(i) For all $(h,k),(h_1,k_1) \in H \times K$, we have $h,h_1 \in H$, $k,k_1 \in K$, so $hh_1 \in H$ and $kk_1 \in K$ as H and K are closed under multiplication, and so $(hh_1,kk_1) \in H \times K$. So we have a binary operation on $H \times K$. In other words, $H \times K$ is closed under multiplication.

(ii) Associativity in $H \times K$ follows from associativity in H and K . For any $(h,k),(h_1,k_1),(h_2,k_2) \in H \times K$, we have

$$\begin{aligned} [(h,k)(h_1,k_1)](h_2,k_2) &= (hh_1,kk_1)(h_2,k_2) \\ &= ((hh_1)h_2,(kk_1)k_2) \\ &= (h(h_1h_2),k(k_1k_2)) \end{aligned}$$

$$\begin{aligned}
&= (h,k)(h_1h_2,k_1k_2) \\
&= (h,k)[(h_1,k_1)(h_2,k_2)]
\end{aligned}$$

and the operation on $H \times K$ is associative.

(iii) What can be the identity element of $H \times K$? The only reasonable guess would be $(1,1) = (1_H, 1_K)$. We indeed have

$$(h,k)(1,1) = (h1,k1) = (h,k)$$

for all $(h,k) \in H \times K$. Thus $(1,1)$ is a right identity of $H \times K$.

(iv) What can be the inverse of $(h,k) \in H \times K$? Probably (h^{-1},k^{-1}) . We indeed have

$$(h,k)(h^{-1},k^{-1}) = (hh^{-1},kk^{-1}) = (1,1)$$

for all $(h,k) \in H \times K$. So any $(h,k) \in H \times K$ has a right inverse in $H \times K$, namely $(h,k)^{-1} = (h^{-1},k^{-1})$.

Therefore, $H \times K$ is a group. □

22.2 Definition: Let H and K be groups. Then the group of Theorem 22.1 is called the *direct product of H and K* . It will be denoted by $H \times K$.

Thus the notation " $H \times K$ " stands for the cartesian product of the sets H and K as well as the direct product of the groups H and K . This ambiguity will not lead to any confusion. The reader should be careful to distinguish between HK and $H \times K$. The former is defined only when H and K are subgroups of a common group G , whereas $H \times K$ is a meaningful group regardless of whether H and K are subgroups of a group. The elements of HK are elements of the group that contains H and K ; the elements of $H \times K$ are ordered pairs.

When the groups H and K are written additively, we write the group of Theorem 22.1 in the additive form, too. The operation is then given by

$$(h,k) + (h_1,k_1) = (h + h_1, k + k_1)$$

for all $(h,k),(h_1,k_1) \in H \times K$. The operation is called *addition* in this case, and the group is called the *direct sum of H and K*. We write the group as $H \oplus K$, to avoid confusion with $H + K$ (which is HK in additive notation, where H and K are subgroups of a group G).

22.3 Examples: (a) Consider $C_2 \times \mathbb{Q}^+$, where $C_2 = \{1,-1\} \subseteq \mathbb{R}$ and \mathbb{Q}^+ is the multiplicative group of the positive rational numbers. The elements of $C_2 \times \mathbb{Q}^+$ are ordered pairs $(\mp 1,q)$, where $q \in \mathbb{Q}^+$. Multiplication in $C_2 \times \mathbb{Q}^+$ is carried out according to the rule $(E,q)(E',q') = (EE',qq')$. We observe that the mapping

$$\begin{aligned} \varphi: \mathbb{Q} \setminus \{0\} &\rightarrow C_2 \times \mathbb{Q}^+ \\ q &\rightarrow (\text{sgn } q, |q|) \end{aligned}$$

is a homomorphism, since

$$\begin{aligned} (qq')\varphi &= (\text{sgn } qq', |qq'|) \\ &= (\text{sgn } q \cdot \text{sgn } q', |q||q'|) \\ &= (\text{sgn } q, |q|)(\text{sgn } q', |q'|) \\ &= (q\varphi)(q'\varphi) \end{aligned}$$

for all $q,q' \in \mathbb{Q} \setminus \{0\}$. Its kernel is

$$\begin{aligned} \text{Ker } \varphi &= \{q \in \mathbb{Q} \setminus \{0\}: q\varphi = (1,1)\} = \{q \in \mathbb{Q} \setminus \{0\}: \text{sgn } q = 1, |q| = 1\} \\ &= \{q \in \mathbb{Q} \setminus \{0\}: q > 0, |q| = 1\} \\ &= \{1\}, \end{aligned}$$

which means that φ is one-to-one (Theorem 20.8). As any $(E,q) \in C_2 \times \mathbb{Q}^+$ is the image of $E|q| \in \mathbb{Q} \setminus \{0\}$, the homomorphism φ is onto. Hence φ is an isomorphism and

$$\mathbb{Q} \setminus \{0\} \cong C_2 \times \mathbb{Q}^+.$$

(b) Consider $\mathbb{R} \oplus \mathbb{R}$, where \mathbb{R} is the additive group of real numbers. The elements of $\mathbb{R} \oplus \mathbb{R}$ are ordered pairs of real numbers. The operation on $\mathbb{R} \oplus \mathbb{R}$ is given by

$$(a,b) + (c,d) = (a + c, b + d)$$

for all $(a,b),(c,d) \in \mathbb{R} \oplus \mathbb{R}$. We leave it to the reader to prove that

$$\begin{aligned} \psi: \mathbb{C} &\rightarrow \mathbb{R} \oplus \mathbb{R} \\ a + bi &\rightarrow (a,b) \end{aligned}$$

is an isomorphism (where \mathbb{C} is the group of complex numbers under addition). Hence

$$\mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}.$$

22.4 Theorem: *Let H and K be groups and let $G := H \times K$ be the direct product of H and K . Then there are subgroups H_1 and K_1 of G such that*

$$\begin{aligned} H_1 &\cong H, & K_1 &\cong K \\ H_1 &\trianglelefteq G, & K_1 &\trianglelefteq G \\ H_1 K_1 &= G, & H_1 \cap K_1 &= 1 \end{aligned}$$

Proof: We put $H_1 = \{(h,1) \in G : h \in H\}$ and $K_1 = \{(1,k) \in G : k \in K\}$. First we prove $H_1, K_1 \leq G$. Since

$$(i) \quad (h,1)(h',1) = (hh',1) \in H_1 \text{ for all } (h,1), (h',1) \in H_1 \text{ and}$$

$$(ii) \quad (h,1)^{-1} = (h^{-1},1^{-1}) = (h^{-1},1) \in H_1 \text{ for all } (h,1) \in H_1,$$

H_1 is a subgroup of G . In the same way, $K_1 \leq G$.

H_1 and K_1 are in fact normal subgroups of G . To establish $K_1 \trianglelefteq G$, we show that $(h,k)^{-1}(1,k_0)(h,k) \in K_1$ for all $(h,k) \in G$, $(1,k_0) \in K_1$ (Lemma 18.2(1)). We indeed have

$$(h,k)^{-1}(1,k_0)(h,k) = (h^{-1},k^{-1})(1,k_0)(h,k) = (h^{-1}1h,k^{-1}k_0k) = (1,k^{-1}k_0k) \in K_1$$

as K is closed under multiplication. Hence $K_1 \trianglelefteq G$. One proves similarly $H_1 \trianglelefteq G$.

Next we show $H \cong H_1$ and $K \cong K_1$. The mapping $\mu_1: H \rightarrow H_1$, $h \rightarrow (h,1)$ is one-to-one (by the definition of equality of ordered pairs) and onto (by the definition of H_1), and is furthermore a homomorphism, since

$$(hh')\mu_1 = (hh',1) = (h,1)(h',1) = h\mu_1 \cdot h'\mu_1$$

for all $h, h' \in H$. Thus μ_1 is an isomorphism and $H \cong H_1$. An analogous argument shows that $\mu_2: K \rightarrow K_1$, $k \rightarrow (1,k)$ is an isomorphism, so $K \cong K_1$.

That $H_1 K_1 = G$ follows immediately from the fact that any $(h,k) \in G$ can be written as $(h,1)(1,k)$ with $(h,1) \in H_1$, $(1,k) \in K_1$.

Finally, $H_1 \cap K_1 = 1$. Indeed, if $(h,k) \in H_1 \cap K_1$, then $h = 1$ as $(h,k) \in K_1$ and $k = 1$ as $(h,k) \in H_1$, thus $(h,k) = (1,1)$ and so $H_1 \cap K_1 \subseteq \{(1,1)\} = 1$, yielding $H_1 \cap K_1 = 1$.

This completes the proof. □

22.5 Theorem: *Let G be a group and let H, K be subgroups of G . The following statements are equivalent.*

(1) $H \triangleleft G, K \triangleleft G, G = HK$ and $H \cap K = 1$.

(2) *Every element of G can be expressed uniquely in the form hk , where $h \in H$ and $k \in K$; and every element of H commutes with every element of K .*

Proof: (1) \Rightarrow (2) Suppose $H \triangleleft G, K \triangleleft G, G = HK$ and $H \cap K = 1$. Since $G = HK$, every element of G can be expressed as hk , with $h \in H, k \in K$. We must show that this representation is unique, i.e., when $hk = h'k'$ with $h, h' \in H$ and $k, k' \in K$, then necessarily $h = h'$ and $k = k'$. This follows from $H \cap K = 1$. Indeed, from $hk = h'k'$, we get $kk^{-1} = h^{-1}h' \in H \cap K = 1$, so $kk^{-1} = 1 = h^{-1}h'$, so $k = k'$ and $h = h'$.

It remains to prove that any element of H commutes with any element of K . Let $h \in H, k \in K$. We have to show $hk = kh$, or, equivalently, $h^{-1}k^{-1}hk = 1$. Now $h^{-1}k^{-1}h.k \in K$, since $h^{-1}k^{-1}h \in K$ (because $k^{-1} \in K$ and $K \triangleleft G$) and $h^{-1}.k^{-1}hk \in H$, since $k^{-1}hk \in H$ (because $h \in H$ and $H \triangleleft G$), so $h^{-1}k^{-1}hk \in H \cap K = 1$ and $h^{-1}k^{-1}hk = 1$, as claimed.

(2) \Rightarrow (1) By hypothesis, every element of G can be written in the form hk , where $h \in H, k \in K$. So $G = HK$. We now prove $H \cap K = 1$. Let $a \in H \cap K$. If $a \neq 1$, then $1a = a1$ are two distinct representations of $a \in G$ with $1 \in H, a \in K$ and $a \in H, 1 \in K$, contrary to the hypothesis that every element of G , in particular a , can be expressed *uniquely* in the form hk , with $h \in H, k \in K$. Thus $a = 1$. This proves $H \cap K = 1$.

In order to prove $H \triangleleft G$, we must show $g^{-1}hg \in H$ for all $h \in H, g \in G$ (Lemma 18.2(1)). Let $g \in G = HK$. Then $g = h'k'$ for some $h' \in H, k' \in K$. Thus

$$\begin{aligned} g^{-1}hg &= (h'k')^{-1}h(h'k') \\ &= k'^{-1}(h'^{-1}hh')k' \\ &= k'^{-1}.k'(h'^{-1}hh') \quad (h'^{-1}hh' \in H \text{ and } k' \in K \text{ commute}) \end{aligned}$$

$$= h'^{-1}hh' \in H$$

and therefore $H \triangleleft G$. The proof of $K \triangleleft G$ is similar and is left to the reader. \square

22.6 Theorem: *Let G be a group and H, K be subgroups of G . Assume that $H \triangleleft G$, $K \triangleleft G$, $G = HK$ and $H \cap K = 1$. Then $G \cong H \times K$.*

Proof: We want to find an isomorphism $\varphi: H \times K \rightarrow G$. For each (h, k) in $H \times K$, this should give us an element $(h, k)\varphi$ of G . By hypothesis, $G = HK$. This suggests that $(h, k) \rightarrow hk$ might be an appropriate mapping from $H \times K$ into G . So we put $\varphi: H \times K \rightarrow G$. We show that φ is a homomorphism,

$$(h, k) \rightarrow hk$$

one-to-one and onto.

φ is a homomorphism if and only if

$$((h', k)(h, k'))\varphi = (h', k)\varphi.(h, k')\varphi \text{ for all } h, h' \in H, k, k' \in K,$$

that is, if and only if

$$h'hkk' = h'khk' \text{ for all } h, h' \in H, k, k' \in K,$$

which is equivalent to

$$hk = kh \text{ for all } h \in H, k \in K,$$

and this is true by Theorem 22.5. So φ is a homomorphism.

φ is one-to-one, for if $(h, k)\varphi = (h', k')\varphi$, then $hk = h'k'$; but every element in G can be expressed in the form hk with $h \in H$, $k \in K$ in a unique way by Theorem 22.5. So $h = h'$ and $k = k'$. Thus $(h, k) = (h', k')$. This proves that φ is one-to-one.

φ is onto because $HK = G$ by hypothesis.

Hence φ is an isomorphism and $H \times K \cong G$, and also $G \cong H \times K$. \square

22.7 Theorem: (1) *A group G is isomorphic to the direct product of two subgroups H and K if and only if (i) every element of G can be expressed uniquely in the form hk , where $h \in H$ and $k \in K$ and (ii) every element of H commutes with every element of K .*

(2) *Let G be a group and $H, K \leq G$. If $G \cong H \times K$, then $G/H \cong K$ and $G/K \cong H$.*

Proof: (1) follows from Theorem 22.4, Theorem 22.5, Theorem 22.6. As for (2), we observe that $G \cong H \times K$ implies $H \triangleleft G$, $K \triangleleft G$, $G = HK$, $H \cap K = 1$, so that $G/H = HK/H \cong K / H \cap K = K/1 \cong K$ by Theorem 21.3. The proof of $G/K \cong H$ is similar. \square

When the conditions of Theorem 22.7(1) are satisfied, G is said to be the internal direct product of H and K . The direct product of Definition 22.2 is called the external direct product of H and K . Theorem 22.5 and Theorem 22.6 state that the internal direct product of H and K is isomorphic to the external direct product $H \times K$. For this reason, we will not distinguish between external and internal direct products and refer to both of them simply as direct products.

As an illustration of Theorem 22.7(1), consider $\mathbb{Q} \setminus \{0\} \cong C_2 \times \mathbb{Q}^+$ (Example 22.3(a)). Theorem 22.7(1) asserts that every nonzero rational number can be written as $(\mp 1)q$, where $q \in \mathbb{Q}$, $q \geq 0$ in a unique way. This is of course well known to everybody.

Next we investigate the direct product of two finite cyclic groups of relatively prime orders. It will be sufficient to examine the direct sum of $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$.

Let m and n be relatively prime natural numbers. For any integer a , we denote the residue class of $a \pmod{m}$ by \bar{a} , and the residue class of $a \pmod{n}$ by a^* . Hence $\bar{a} \in \mathbb{Z}_m$ and $a^* \in \mathbb{Z}_n$.

Consider the mapping $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_m \oplus \mathbb{Z}_n$. It is easy to see that φ is a homo-

$$a \rightarrow (\bar{a}, a^*)$$

morphism:

$$(a+b)\varphi = (\overline{a+b}, (a+b)^*) = (\bar{a} + \bar{b}, a^* + b^*) = (\bar{a}, a^*) + (\bar{b}, b^*) = a\varphi + b\varphi$$

for all $a, b \in \mathbb{Z}$. So φ is a homomorphism and

$$\mathbb{Z}/\text{Ker } \varphi \cong \text{Im } \varphi$$

by Theorem 20.16. Now $a \in \text{Ker } \varphi$ if and only if $\bar{a} = \bar{0}$ and $a^* = 0^*$, that is, if and only if $m|a$ and $n|a$. Since m and n are relatively prime, the latter condition is equivalent to $mn|a$. Hence $\text{Ker } \varphi = mn\mathbb{Z}$ and $\mathbb{Z}/mn\mathbb{Z} \cong \text{Im } \varphi$, where $\text{Im } \varphi$ is a subgroup of $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. From

$$mn = |\mathbb{Z}/mn\mathbb{Z}| = |\text{Im } \varphi| \leq |\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}| = |\mathbb{Z}/m\mathbb{Z}| |\mathbb{Z}/n\mathbb{Z}| = mn$$

we conclude $|\text{Im } \varphi| = mn$, hence $\text{Im } \varphi = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Therefore φ is onto and $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Writing this multiplicatively, we get

22.8 Theorem: *If m and n are relatively prime natural numbers, then*

$$C_{mn} \cong C_m \times C_n. \quad \square$$

We record an important result that we obtained as a bonus.

22.9 Theorem: *Let m and n be relatively prime natural numbers. Then the mapping*

$$\begin{aligned} \varphi: \mathbb{Z} &\rightarrow \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \\ a &\rightarrow (\bar{a}, a^*) \end{aligned}$$

is a group homomorphism onto $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. □

So far, we have examined the direct product of two groups. The construction extends immediately to n groups, where $n > 2$. We shall be content with enunciating the appropriate theorems. Their proofs consist in writing n -tuples in place of ordered pairs in the proofs above. The only novel point is extension of the previous condition $H \cap K = 1$. This is discussed in Theorem 22.12, whose proof we briefly sketch.

22.10 Theorem: *Let H_1, H_2, \dots, H_n be arbitrary groups. On the cartesian product $H_1 \times H_2 \times \dots \times H_n$, we define a binary operation by declaring*

$$(h_1, h_2, \dots, h_n)(h_1', h_2', \dots, h_n') = (h_1 h_1', h_2 h_2', \dots, h_n h_n')$$

for all $(h_1, h_2, \dots, h_n), (h_1', h_2', \dots, h_n') \in H_1 \times H_2 \times \dots \times H_n$. With respect to this operation, $H_1 \times H_2 \times \dots \times H_n$ is a group.

22.11 Definition: The group of Theorem 22.10 is called the *direct product* of H_1, H_2, \dots, H_n and is denoted by $H_1 \times H_2 \times \dots \times H_n$. If the groups are written additively, we call the group of Theorem 22.10 the *direct sum* of H_1, H_2, \dots, H_n and denote it by $H_1 \oplus H_2 \oplus \dots \oplus H_n$.

22.12 Theorem: Let H_1, H_2, \dots, H_n be groups and $G = H_1 \times H_2 \times \dots \times H_n$. Then there are subgroups G_1, G_2, \dots, G_n of G such that

$$G_i \cong H_i \text{ and } G_i \triangleleft H_i \text{ for all } i = 1, 2, \dots, n,$$

$$G = G_1 G_2 \dots G_n \text{ and } G_1 G_2 \dots G_{j-1} \cap G_j = 1 \text{ for all } j = 2, \dots, n.$$

Sketch of proof: Let G_i be the set $\{(1, \dots, x, \dots, 1) : x \in H_i\}$ of all n -tuples in G whose k -th components are equal to $1 \in H_k$ whenever $k \neq i$. It is easily verified that G_i is a subgroup of G , normal in G , isomorphic to H_i and that $G = G_1 G_2 \dots G_n$. In fact, for all $j = 2, \dots, n$,

$$G_1 G_2 \dots G_{j-1} = \{(h_1, h_2, \dots, h_{j-1}, 1, \dots, 1) : h_1 \in H_1, h_2 \in H_2, \dots, h_{j-1} \in H_{j-1}\}.$$

Finally, to prove $G_1 G_2 \dots G_{j-1} \cap G_j = 1$ for all $j = 2, \dots, n$, let

$(u_1, u_2, \dots, u_n) \in G_1 G_2 \dots G_{j-1} \cap G_j$, where $j \in \{2, \dots, n\}$. Here $u_k = 1$ for $k \neq j$, because $(u_1, u_2, \dots, u_n) \in G_j$. Thus $(u_1, u_2, \dots, u_n) = (1, \dots, u_j, \dots, 1)$. But

$$(1, \dots, u_j, \dots, 1) \in G_1 G_2 \dots G_{j-1} \\ = \{(h_1, h_2, \dots, h_{j-1}, 1, \dots, 1) : h_1 \in H_1, h_2 \in H_2, \dots, h_{j-1} \in H_{j-1}\},$$

hence $u_j = 1$ and

$$(u_1, u_2, \dots, u_n) = (1, \dots, 1, \dots, 1) = 1 \in G. \text{ Thus } G_1 G_2 \dots G_{j-1} \cap G_j = 1. \quad \square$$

22.13 Theorem: Let G be a group and let G_1, G_2, \dots, G_n be subgroups of G . The following statements are equivalent.

(1) $G_i \triangleleft H_i$ for all $i = 1, 2, \dots, n$, $G = G_1 G_2 \dots G_n$ and $G_1 G_2 \dots G_{j-1} \cap G_j = 1$ for all $j = 2, \dots, n$.

(2) Every element of G can be expressed uniquely in the form $g_1 g_2 \dots g_n$, where $g_1 \in G_1, g_2 \in G_2, \dots, g_n \in G_n$; and every element of G_k commutes with every element of G_l ($k \neq l$).

22.14 Theorem: Let G be a group and let G_1, G_2, \dots, G_n be subgroups of G . Assume that $G_i \triangleleft G$ for all $i = 1, 2, \dots, n$, $G = G_1 G_2 \dots G_n$ and $G_1 G_2 \dots G_{j-1} \cap G_j = 1$ for all $j = 2, \dots, n$. Then $G \cong G_1 \times G_2 \times \dots \times G_n$.

If $n > 3$ and G_1, G_2, \dots, G_n are normal subgroups of a group G such that $G = G_1 G_2 \dots G_n$ and $G_i \cap G_j = 1$ whenever $i \neq j$, then G is need not be isomorphic to the direct product of G_1, G_2, \dots, G_n . By way ow example, let $G = V_4$ and let $A = \{1, (12)(34)\}$, $B = \{1, (13)(24)\}$, $C = \{1, (23)(14)\}$. Then A, B, C are normal subgroups of G , and $G = ABC$, and $A \cap B = B \cap C = A \cap C = 1$. However, G is not isomorphic to $A \times B \times C$, because, for one thing, G has order 4, whereas $A \times B \times C$ has order 8. Thus the condition

$$G_1 G_2 \dots G_{j-1} \cap G_j = 1 \text{ for all } j = 2, \dots, n$$

cannot be relaxed to

$$G_i \cap G_j = 1 \text{ for all } i \neq j.$$

22.15 Theorem: A group G is isomorphic to the direct product of n sub-groups G_1, G_2, \dots, G_n if and only if (i) every element of G can be expressed uniquely in the form $g_1 g_2 \dots g_n$, where $g_1 \in G_1, g_2 \in G_2, \dots, g_n \in G_n$ and (ii) every element of G_k commutes with every element of G_l ($k \neq l$).

The last two elementary results will be needed in §28.

22.16 Lemma: Let $G_1, G_2, \dots, G_n, H_1, H_2, \dots, H_n$ be groups and assume that $G_1 \cong H_1, G_2 \cong H_2, \dots, G_n \cong H_n$. Then $G_1 \times G_2 \times \dots \times G_n \cong H_1 \times H_2 \times \dots \times H_n$.

Proof: Let $\varphi_i: G_i \rightarrow H_i$ be an isomorphism ($i = 1, 2, \dots, n$). The mapping

$$\begin{aligned} \psi: G_1 \times G_2 \times \dots \times G_n &\rightarrow H_1 \times H_2 \times \dots \times H_n \\ (g_1, g_2, \dots, g_n) &\rightarrow (g_1 \varphi_1, g_2 \varphi_2, \dots, g_n \varphi_n) \end{aligned}$$

is a homomorphism, because $((g_1, g_2, \dots, g_n)(g_1', g_2', \dots, g_n'))\psi$

$$= (g_1 g_1', g_2 g_2', \dots, g_n g_n')\psi$$

$$= ((g_1 g_1')\varphi_1, (g_2 g_2')\varphi_2, \dots, (g_n g_n')\varphi_n)$$

$$= (g_1 \varphi_1 g_1' \varphi_1, g_2 \varphi_2 g_2' \varphi_2, \dots, g_n \varphi_n g_n' \varphi_n)$$

$$= (g_1 \varphi_1, g_2 \varphi_2, \dots, g_n \varphi_n)(g_1' \varphi_1, g_2' \varphi_2, \dots, g_n' \varphi_n)$$

$$= (g_1, g_2, \dots, g_n) \psi (g_1', g_2', \dots, g_n') \psi$$

for all $(g_1, g_2, \dots, g_n), (g_1', g_2', \dots, g_n') \in G_1 \times G_2 \times \dots \times G_n$. Since

$$\begin{aligned} \text{Ker } \psi &= \{(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n : (g_1 \phi_1, g_2 \phi_2, \dots, g_n \phi_n) = (1, 1, \dots, 1)\} \\ &= \{(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n : g_1 \phi_1 = 1, g_2 \phi_2 = 1, \dots, g_n \phi_n = 1\} \\ &= \{(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n : g_1 = 1, g_2 = 1, \dots, g_n = 1\} \\ &= \{(1, 1, \dots, 1)\} = 1, \end{aligned}$$

ψ is one-to-one. Also, ψ is onto: given any $(h_1, h_2, \dots, h_n) \in H_1 \times H_2 \times \dots \times H_n$, there are $g_1 \in G_1, g_2 \in G_2, \dots, g_n \in G_n$ with $g_1 \phi_1 = h_1, g_2 \phi_2 = h_2, \dots, g_n \phi_n = h_n$, thus (h_1, h_2, \dots, h_n) is the image, under ψ , of $(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n$.

So ψ is an isomorphism and $G_1 \times G_2 \times \dots \times G_n \cong H_1 \times H_2 \times \dots \times H_n$. \square

22.17 Lemma: Let G_1, G_2, \dots, G_n be groups and $H_1 \trianglelefteq G_1, H_2 \trianglelefteq G_2, \dots, H_n \trianglelefteq G_n$.

Then $H_1 \times H_2 \times \dots \times H_n \trianglelefteq G_1 \times G_2 \times \dots \times G_n$ and

$$G_1 \times G_2 \times \dots \times G_n / H_1 \times H_2 \times \dots \times H_n \cong G_1/H_1 \times G_2/H_2 \times \dots \times G_n/H_n.$$

Proof: The mapping $\varphi: G_1 \times G_2 \times \dots \times G_n \longrightarrow G_1/H_1 \times G_2/H_2 \times \dots \times G_n/H_n$
 $(g_1, g_2, \dots, g_n) \longrightarrow (H_1 g_1, H_2 g_2, \dots, H_n g_n)$

is a homomorphism because $((g_1, g_2, \dots, g_n)(g_1', g_2', \dots, g_n')) \varphi$

$$\begin{aligned} &= (g_1 g_1', g_2 g_2', \dots, g_n g_n') \varphi \\ &= (H_1 g_1 g_1', H_2 g_2 g_2', \dots, H_n g_n g_n') \\ &= (H_1 g_1 H_1 g_1', H_2 g_2 H_2 g_2', \dots, H_n g_n H_n g_n') \\ &= (H_1 g_1, H_2 g_2, \dots, H_n g_n) (H_1 g_1', H_2 g_2', \dots, H_n g_n') \\ &= (g_1, g_2, \dots, g_n) \varphi (g_1', g_2', \dots, g_n') \varphi \end{aligned}$$

for all $(g_1, g_2, \dots, g_n), (g_1', g_2', \dots, g_n') \in G_1 \times G_2 \times \dots \times G_n$. Moreover, φ is onto: any $(H_1 g_1, H_2 g_2, \dots, H_n g_n)$ in $G_1/H_1 \times G_2/H_2 \times \dots \times G_n/H_n$ is the image, under φ , of $(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n$. Thus

$$\text{Im } \varphi = G_1/H_1 \times G_2/H_2 \times \dots \times G_n/H_n.$$

To complete the proof, we need only show $\text{Ker } \varphi = H_1 \times H_2 \times \dots \times H_n$ (Theorem 20.16). We indeed have

$$\begin{aligned}
\text{Ker } \varphi &= \{(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n : (g_1, g_2, \dots, g_n)\varphi = (H_1, H_2, \dots, H_n)\} \\
&= \{(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n : H_1 g_1 = H_1, H_2 g_2 = H_2, \dots, H_n g_n = H_n\} \\
&= \{(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n : g_1 \in H_1, g_2 \in H_2, \dots, g_n \in H_n\} \\
&= H_1 \times H_2 \times \dots \times H_n. \quad \square
\end{aligned}$$

Exercises

1. Prove that $V_4 \cong C_2 \times C_2$.
2. Show that C_{mn} is not isomorphic to $C_m \times C_n$ if $(m, n) \neq 1$.
3. Find three nonisomorphic abelian groups of order 8 and three nonisomorphic abelian groups of order 12.
4. Show that $G_1 \times G_2 \times \dots \times G_n \cong G_{i_1} \times G_{i_2} \times \dots \times G_{i_n}$ for any permutation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \text{ in } S_n.$$

5. Prove that, if G is isomorphic to the direct product of its subgroups G_1, G_2, \dots, G_n , then $G_1 \dots G_{k-1} G_{k+1} \dots G_n \cap G_k = 1$ for all $k = 1, 2, \dots, n$.
6. Let H, K be normal subgroups of G . Find a one-to-one homomorphism from $G/H \cap K$ into $G/H \times G/K$. Prove that $HK/H \cap K \cong H/H \cap K \times K/H \cap K$.
7. Let $\varphi_i: G_i \rightarrow H_i$ be group homomorphisms ($i = 1, 2, \dots, n$). Define ψ by

$$\begin{aligned}
\psi: G_1 \times G_2 \times \dots \times G_n &\rightarrow H_1 \times H_2 \times \dots \times H_n \\
(g_1, g_2, \dots, g_n) &\rightarrow (g_1 \varphi_1, g_2 \varphi_2, \dots, g_n \varphi_n)
\end{aligned}$$

(ψ is sometimes denoted by $\varphi_1 \times \varphi_2 \times \dots \times \varphi_n$). Show that ψ is a homomorphism and

$$\text{Ker } \psi = \text{Ker } \varphi_1 \times \text{Ker } \varphi_2 \times \dots \times \text{Ker } \varphi_n, \quad \text{Im } \psi = \text{Im } \varphi_1 \times \text{Im } \varphi_2 \times \dots \times \text{Im } \varphi_n.$$

8. For any abelian group A , let \hat{A} be the set of all homomorphisms from A into $\mathbb{C} \setminus \{0\}$. Prove that \hat{A} is an abelian group under the multiplication

$$a(\varphi\psi) = a\varphi.a\psi \quad \text{for all } a \in A, \varphi, \psi \in \hat{A}$$

and show that $\hat{A}_1 \times \hat{A}_2 \cong A_1 \times A_2$.