§23
Center and Automorphisms of Groups

We introduce an important subgroup of a group.

23.1 Definition: Let G be a group. We put

$$Z(G) = \{z \in G: zg = gz \text{ for all } g \in G\}$$

and call $Z(G)$ the *center of G*.

The center of G consists, therefore, of the elements of G that commute with every element of G. It is a subset of G. Since $1g = g1$ for all $g \in G$, the identity element belongs to $Z(G)$, so $Z(G) \neq \varnothing$. Obviously, $Z(G) = G$ if and only if G is abelian.

23.2 Theorem: *Let G be a group. Then $Z(G) \leqslant G$.*

Proof: We use our subgroup criterion (Lemma 9.2).

(i) Let $z_1,z_2 \in Z(G)$. We want to show $z_1z_2 \in Z(G)$. Thus we must show that $(z_1z_2)g = g(z_1z_2)$ for all $g \in G$. This follows easily from $z_1g = gz_1$, $z_2g = gz_2$ for all $g \in G$, which are true since $z_1,z_2 \in Z(G)$:

$$(z_1z_2)g = z_1(z_2g) = z_1(gz_2) = (z_1g)z_2 = (gz_1)z_2 = g(z_1z_2).$$

Hence $Z(G)$ is closed under multiplication.

(ii) Let $z \in Z(G)$. We want to show $z^{-1} \in Z(G)$. We know

$$g^{-1}z = zg^{-1} \text{ for any } g \in G;$$

so, taking inverses, we get

$$z^{-1}g = gz^{-1} \text{ for any } g \in G,$$

which means $z^{-1} \in Z(G)$. Hence $Z(G)$ is closed under the forming of inverses.

Thus $Z(G) \leqslant G$. □

As any two elements of $Z(G)$ commute, $Z(G)$ is an abelian subgroup of G. It is also a normal subgroup of G. We prove a slightly stronger result.

23.3 Theorem: *Let G be a group. If* $H \leqslant Z(G)$, *then* $H \trianglelefteq G$.

Proof: We are to show $g^{-1}hg \in H$ for all $g \in G$, $h \in H$. Now, if $g \in G$, $h \in H$ then $g^{-1}hg = g^{-1}(hg) = g^{-1}(gh) = (g^{-1}g)h = h \in H$, for $h \in Z(G)$ commutes with g. Thus $H \trianglelefteq G$. □

A subgroup of G which is contained in the center of G is called a *central subgroup of G*. With this terminology, Theorem 23.3 states that any central subgroup of G is normal in G. Central subgroups are abelian. Elements of $Z(G)$ are also called *central elements of G*.

23.4 Examples: (a) Let K be a field and let us put $G = GL(2,K)$ for brevity. We want to find $Z(G)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(G)$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} x & y \\ z & u \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for all $\begin{pmatrix} x & y \\ z & u \end{pmatrix} \in G$. In particular,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

hence

$$\begin{array}{llcll} a = a + c, & a + b = b + d & \text{and} & b = c & a = d \\ c = c, & c + d = d & & d = a & c = b \end{array}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(G)$, so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, where $a \neq 0$ since $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Therefore $$Z(G) \subseteq \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G : a \neq 0\right\}$$

and conversely the set on the right hand side is contained in $Z(G)$, for

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & au \end{pmatrix} = \begin{pmatrix} xa & ya \\ za & ua \end{pmatrix} = \begin{pmatrix} x & y \\ z & u \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ for all } \begin{pmatrix} x & y \\ z & u \end{pmatrix} \in G.$$

Thus $Z(G) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G : a \neq 0\right\}$. The elements of $Z(G)$ are called *scalar* matrices.

(b) Let $D_{4n} = \langle a,b\colon a^{2n} = 1, b^2 = 1, bab = a^{-1}\rangle$ be a dihedral group of order $4n > 4$. What is $Z(D_{4n})$? Well, let $x \in Z(D_{4n})$. Then $x = a^j$ or $x = a^jb$ for some $j \in \mathbb{Z}$, $0 \leqslant j \leqslant 2n - 1$. Since $xa = ax$ and $xb = bx$, we get

$a^ja = aa^j$	and	$a^jb = ba^j$	in case $x = a^j$,
$a^jba = aa^jb$	and	$a^jbb = ba^jb$	in case $x = a^jb$.

These are equivalent to

(1)	$a^{j+1} = a^{j+1}$	and	$a^jb = a^{-j}b$	in case $x = a^j$,
(2)	$a^{j-1}b = a^{j+1}b$	and	$a^j = a^{-j}$	in case $x = a^jb$.

The equations in (1) are satisfied only when $n|2j$, that is to say, only when $j = 0,n$, so only when $x = a^0,a^n$. The first equation in (2) is never satisfied, for $n > 1$ by hypothesis. Thus $Z(D_{4n}) \subseteq \{1,a^n\}$. The reader will easily show the reverse inclusion. Hence $Z(D_{4n}) = \{1,a^n\} = \langle a^n\rangle$.

(c) Let us find $Z(S_3)$. It is easy to see that ι and (12) are the only permutations in S_3 that commute with (12). Also, ι and (13) are the only permu-tations in S_3 that commute with (13). Hence ι is the only permutation in S_3 that commute with both (12) and (13). A fortiori, $Z(S_3) = 1$.

23.5 Lemma: *Let H be a central subgroup of G. If G/H is cyclic, then G is abelian.*

Proof: $H \trianglelefteq G$ by Theorem 23.3 and so G/H is meaningful. By hypothesis, $G/H = \langle Hg\rangle$ for some $g \in G$. Then, for any $x \in G$, there holds $Hx = (Hg)^m = Hg^m$ with a suitable $m \in \mathbb{Z}$. This means that any $x \in G$ can be written in the form hg^m, where $h \in H$, $m \in \mathbb{Z}$.

Let x,y be arbitrary elements of G. We write them as $x = hg^m$, $y = kg^n$, where $h,k \in H \leqslant Z(G)$ and $m,n \in \mathbb{Z}$. Then $xy = (hg^m)(kg^n) = h(g^mk)g^n = h(kg^m)g^n = (hk)(g^mg^n) = (hk)(g^{m+n}) = (hk)(g^{n+m}) = (kh)(g^ng^m) = k(hg^n)g^m = k(g^nh)g^m = (kg^n)(hg^m) = yx$ and G is commutative. □

The center of any group G is normal in G (Theorem 23.3) and is therefore the kernel of some homomorphism (Theorem 20.14). Now we construct a homomorphism whose kernel is $Z(G)$. We will need the concept of auto-morphisms.

23.6 Definition: Let G be a group. An isomorphism $\alpha{:}G \to G$ from G onto G itself is called an *automorphism of G*. The set of all automorphisms of G will be denoted by $Aut(G)$.

Since any isomorphism is one-to-one and onto, $\alpha \in Aut(G)$ implies $\alpha \in S_G$. Thus $Aut(G) \subseteq S_G$. The identity mapping ι_G on G is an isomorphism from G onto G, so $\iota_G \in Aut(G)$ and $Aut(G) \neq \emptyset$. We can form the composition $\alpha\beta$ of any $\alpha,\beta \in Aut(G)$. It turns out that $Aut(G)$ is a group.

23.7 Theorem: *Let G be a group. Then* $Aut(G)$ *is a group under the composition of mappings.*

Proof: We can check the group axioms, but there is a shorter way. We make use of $\emptyset \neq Aut(G) \subseteq S_G$. Now S_G is a group under the composition of mappings (Example 7.1(d)), so all we have to do is show that $Aut(G)$ is a subgroup of S_G.

(i) Let $\alpha,\beta \in Aut(G)$. Then $\alpha\beta$ is an isomorphism from G onto G by Lemma 20.11(1). Thus $\alpha\beta \in Aut(G)$ and $Aut(G)$ is closed under multiplication.

(ii) Let $\alpha \in Aut(G)$. Then α^{-1} is an isomorphism from G onto G by Lemma 20.11(2). Thus $\alpha^{-1} \in Aut(G)$ and $Aut(G)$ is closed under the forming of inverses.

By Lemma 9.2, $Aut(G) \leqslant S_G$. Thus $Aut(G)$ is a group. □

$Aut(G)$ is not a subgroup or a factor group of G, of course. The underlying set is neither a subset nor a set of cosets of a subgroup of G.

23.8 Example: Let G be a group. We fix an arbitrary element g of G. With each $x \in G$, we associate $g^{-1}xg$. This is a uniquely determined element of G, so we have a mapping $x \to g^{-1}xg$, which we denote by τ_g. So

$$\begin{aligned} \tau_g \colon G &\to G \\ x &\to g^{-1}xg \end{aligned}$$

We claim τ_g is a homomorphism. For all $x,y \in G$, we have

$$(xy)\tau_g = g^{-1}xyg = g^{-1}xg.g^{-1}yg = x\tau_g y\tau_g,$$

and τ_g is therefore a homomorphism.

We can build τ_g with any $g \in G$. Let us take the composition of two of them, τ_g and τ_h, say. For $g,h \in G$, we have

$$x(\tau_g\tau_h) = (x\tau_g)\tau_h = (g^{-1}xg)\tau_h = h^{-1}(g^{-1}xg)h = (h^{-1}g^{-1})x(gh) = (gh)^{-1}x(gh) = x\tau_{gh}$$

for all $x \in G$. Thus

$$\tau_g\tau_h = \tau_{gh} \qquad \text{for all } g,h \in G.$$

(1)

There holds $x\tau_1 = 1^{-1}x1 = x$ for all $x \in G$. Thus

$$\tau_1 = \iota.$$

(2)

For any $g \in G$, there holds $\tau_g\tau_{g^{-1}} = \tau_{gg^{-1}} = \tau_1 = \iota$ and $\tau_{g^{-1}}\tau_g = \tau_{g^{-1}g} = \tau_1 = \iota$ by (1) and (2). Thus τ_g is one-to-one and onto (Theorem 3.17(2)) and $\tau_{g^{-1}}$ is the inverse of τ_g:

$$(\tau_g)^{-1} = \tau_{g^{-1}} \tag{3}$$

So τ_g is an automorphism of G.

Such automorphisms deserve a name.

23.9 Definition: Let G be a group. An automorphism of G of the form τ_g, where $g \in G$, is called an *inner automorphism of* G. The set

$$\{\tau_g \in Aut(G) \colon g \in G\}$$

of all inner automorphisms of G will be denoted by $Inn(G)$.

Inner automorphisms of a group form a group.

23.10 Theorem: *Let G be a group. Then $Inn(G) \leqslant Aut(G)$.*

Proof: $\iota = \tau_1 \in Inn(G)$ by (2), so $Inn(G) \neq \varnothing$. Now (i) the product of two inner automorphisms is an inner automorphism by (1); and (ii) the inverse of an inner automorphism is an inner automorphism by (3). So $Inn(G) \leqslant Aut(G)$. □

The relation (1) has a deep significance. It states that the mapping

$$\begin{aligned} \tau\colon G &\to Aut(G) \\ g &\to \tau_g \end{aligned}$$

is a homomorphism. Theorem 20.16 gives $G/Ker\ \tau \cong Im\ \tau$.

Here $Im\ \tau = \{\tau_g \in Aut(G) \colon g \in G\} = Inn(G)$ by definition and

$$\begin{aligned} Ker\ \tau &= \{z \in G \colon \tau_z = \iota\} \\ &= \{z \in G \colon g\tau_z = g \text{ for all } g \in G\} \\ &= \{z \in G \colon z^{-1}gz = g \text{ for all } g \in G\} \\ &= \{z \in G \colon gz = zg \text{ for all } g \in G\} \\ &= Z(G). \end{aligned}$$

Thus $Z(G)$ is the kernel of $\tau\colon G \to Aut(G)$. We proved

23.11 Theorem: *Let G be a group. Then $G/Z(G) \cong Inn(G)$.* □

Next we prove that $Inn(G)$ is a normal subgroup of $Aut(G)$.

23.12 Lemma: *Let G be a group. Then $Inn(G) \trianglelefteq Aut(G)$.*

Proof: We know $Inn(G) \leqslant Aut(G)$ from Theorem 23.10. We are to show $\sigma^{-1}\tau_g\sigma \in Inn(G)$ for any $\tau_g \in Inn(G)$, $\sigma \in Aut(G)$. For any $x \in G$, we have

$$\begin{aligned} x(\sigma^{-1}\tau_g\sigma) &= (x\sigma^{-1})(\tau_g\sigma) \\ &= \big((x\sigma^{-1})\tau_g\big)\sigma \\ &= \big(g^{-1}(x\sigma^{-1})g\big)\sigma \\ &= (g^{-1}\sigma)\big((x\sigma^{-1})\sigma\big)(g\sigma) \\ &= (g\sigma)^{-1}x(g\sigma) \\ &= x\tau_{g\sigma}, \end{aligned}$$

thus $\sigma^{-1}\tau_g\sigma = \tau_{g\sigma}$ and $\sigma^{-1}\tau_g\sigma \in Inn(G)$. This proves $Inn(G) \trianglelefteq Aut(G)$. □

Let G be a group and let $H \trianglelefteq G$. According to Lemma 18.2(3), $H \trianglelefteq G$ if and only if $H\tau_g = H$ for all $\tau_g \in Inn(G)$. This suggests a way of strengthening the normality concept: Instead of requiring $H\sigma = H$ for all $\sigma \in Inn(G)$, we prescribe this to hold for all $\sigma \in Aut(G)$.

23.13 Definition: Let G be a group. A subgroup H of is said to be a *characteristic subgroup of* G or to be *characteristic in G* provided $H\sigma = H$ for all $\sigma \in Aut(G)$.

Here $H\sigma$ means the set $\{h\sigma\colon h \in H\} \subseteq G$ as usual. The equality $H\sigma = H$ is a set equality, of course. It does *not* mean that $h\sigma = h$ for all $h \in H$. It means that, $h\sigma \in H$ for any $h \in H$, and, for any $h \in H$, there is an $h' \in H$ such that $h'\sigma = h$. Cf. Example 18.5(b). As $Inn(G) \leqslant Aut(G)$, any characteristic subgroup of G is normal in G, but the converse is not true in general.

Being characteristic is a transitive relation, a good property not shared by normality (Example 18.5(i)).

23.14 Lemma: *Let $K \leqslant H \leqslant G$. If K is characteristic in H and H is characteristic in G, then K is characteristic in G.*

Proof: We are to prove that $K\sigma = K$ for all $\sigma \in Aut(G)$. Let $\sigma \in Aut(G)$. We restrict σ to H. Then $\sigma_H\colon H \to G$ is a one-to-one homomorphism onto $H\sigma$. Since H is characteristic in G,we have $H\sigma = H$ and σ_H is an automorphism of H. Then $K\sigma_H = K$, because K is characteristic in H. Thus $K\sigma = K$ for all σ in $Aut(G)$ and K is a characteristic subgroup of G. □

Another useful result of this type is given in the next lemma.

23.15 Lemma: *Let $K \leqslant H \leqslant G$. If K is characteristic in H and H is normal in G, then K is normal in G.*

Proof: We are to prove that $K\tau_g = K$ for all $\tau_g \in Inn(G)$. Let $\tau_g \in Inn(G)$. We restrict τ_g to H. Then $\tau_{g|H}\colon H \to G$ is a one-to-one homomorphism onto $H\tau_g = g^{-1}Hg$. Since H is normal in G,we have $g^{-1}Hg = H$ and $\tau_{g|H}$ is an automorphism of H. Then $K\tau_{g|H} = K$, because K is characteristic in H. Thus $g^{-1}Kg = K\tau_{g|H} = K$ for all $g \in G$ and K is a normal subgroup of G. □

23.16 Theorem: *Let G be a group. Then $Z(G)$ is characteristic in G.*

Proof: We must show $Z(G)\sigma = Z(G)$ for all $\sigma \in Aut(G)$. If we can prove $Z(G)\sigma \subseteq Z(G)$ for all $\sigma \in Aut(G)$, then we will have $Z(G)\sigma^{-1} \subseteq Z(G)$, that is, $Z(G) \subseteq Z(G)\sigma$ for any $\sigma \in Aut(G)$ also (cf. the proof of (2) $\Rightarrow$ (3) in Lemma 18.2). So we need only prove $Z(G)\sigma \subseteq Z(G)$. For any $z \in Z(G)$, we are to show that $(z\sigma)g = g(z\sigma)$ for all $g \in G$. As g runs through G, so does $g\sigma$, because σ is *onto* G. Thes we need only show $(z\sigma)(g\sigma) = (g\sigma)(z\sigma)$ for all $g \in G$. But this is obvious: $(z\sigma)(g\sigma) = (zg)\sigma = (gz)\sigma = (g\sigma)(z\sigma)$ since $z \in Z(G)$ and σ is a homomorphism. Consequently, $Z(G)$ is characteristic in G. □

We end this paragraph by finding the automorphism group of a finite cyclic group. In general, given a group G, it is quite difficult to find $Aut(G)$.

Let $C_n = \langle x \colon x^n = 1\rangle$ be a cyclic group of order $n \in \mathbb{N}$. An automorphism of C_n is first of all a homomorphism of C_n. We claim that a homomorphism from C_n into C_n is uniquely determined by its effect on the generator x. In other words, if α and β are homomorphisms from C_n into C_n and $x\alpha = x\beta$, then $\alpha = \beta$. To show this, we must prove $a\alpha = a\beta$ for all $a \in C_n$. But $a =$

x^m for some $m \in \mathbb{N}$, and $a\alpha = x^m\alpha = (x\alpha)^m = (x\beta)^m = x^m\beta = a\beta$. This proves the claim.

Let ψ be a homomorphism from C_n into C_n. Then $x\psi = x^m$ for some $m \in \mathbb{Z}$. Then $x^k\psi = (x\psi)^k = (x^m)^k = x^{mk} = (x^k)^m$ for any $k \in \mathbb{N}$. This shows $a\psi = a^m$ for any $a \in C_n$. Thus a homomorphism from C_n into C_n simply sends each element of C_n to its m-th power, m being a natural number depending only on the homomorphism. The homomorphism of taking m-th powers will be denoted by α_m. Hence

$$\begin{aligned}\alpha_m\colon \langle x\rangle &\to \langle x\rangle\\ a &\to a^m\end{aligned}$$

is a homomorphism from C_n into C_n, and any homomorphism from C_n into C_n is one of the α_m.

From the homomorphisms $\{\alpha_m : m \in \mathbb{Z}\}$, we want to select the homomorphisms. These are the one-to-one α_m's onto C_n. Since C_n is a finite set, any one-to-one mapping from C_n into C_n is in fact onto C_n. So we need find only one-to-one α_m's. These and exactly these are the automorphisms of C_n.

Now α_m is one-to-one if and only if $Ker\ \alpha_m = 1$ (Theorem 20.8) and

$$\begin{aligned}Ker\ \alpha_m &= \{g \in C_n : g\alpha_m = 1\}\\ &= \{x^k : k \in \mathbb{Z} \text{ and } x^{km} = 1\}\\ &= \{x^k : k \in \mathbb{Z} \text{ and } n|km\}\\ &= \{x^k : k \in \mathbb{Z} \text{ and } n/(n,m) \mid km/(n,m)\}\\ &= \{x^k : k \in \mathbb{Z} \text{ and } n/(n,m) \mid k\}\\ &= \langle x^{n/(n,m)}\rangle,\end{aligned}$$

so $Ker\ \alpha_m = 1 = \langle x^n\rangle$ if and only if $(n,m) = 1$. Thus α_m is an automorphism of C_n if and only if $(n,m) = 1$.

Hence $$Aut(C_n) = \{\alpha_m : (n,m) = 1\}.$$

This description of $Aut(C_n)$ looks like an infinite set. $Aut(C_n)$ is finite of course. Therefore, there are repetitions among α_m. To see this more vividly, we remark that $\alpha_m = \alpha_k$ if and only if $m \equiv k \pmod{n}$. Indeed, α_m is equal to α_k if and only if $x\alpha_m = x\alpha_k$ by the claim above, thus if and only if $x^m = x^k$, thus if and only if $x^{m-k} = 1$, thus if and only if $n \mid m - k$ by Lemma 11.6, thus if and only if $m \equiv k \pmod{n}$.

Hence, for any $\overline{m} \in \mathbb{Z}_n$, we may unambiguously write $\alpha_{\overline{m}} \colon C_n \to C_n$. With

$a \to a^m$

this notation, we have

$$Aut(C_n) = \{\alpha_{\overline{m}} : \overline{m} \in \mathbb{Z}_n\}$$

and $\overline{m} \neq \overline{k}$ implies $\alpha_{\overline{m}} \neq \alpha_{\overline{k}}$. In other words, the mapping

$$\alpha\colon \mathbb{Z}_n^{\times} \to Aut(C_n)$$

is one-to-one and onto. It is a homomorphism, because

$$x\alpha_{\overline{mk}} = x^{mk} = (x^m)^k = (x^m)\alpha_{\overline{k}} = (x\alpha_{\overline{m}})\alpha_{\overline{k}} = x(\alpha_{\overline{m}}\alpha_{\overline{k}})$$

and, by the claim at the beginning, $\alpha_{\overline{mk}} = \alpha_{\overline{m}}\alpha_{\overline{k}}$ for any $\overline{m}, \overline{k} \in \mathbb{Z}_n$. Hence α is an isomorphism and $\mathbb{Z}_n^{\times} \cong Aut(C_n)$. We proved

23.17 Theorem: *If G is a cyclic group of order $n \in \mathbb{N}$, then $Aut(G) \cong \mathbb{Z}_n^{\times}$.*

□

Exercises

1. Let H,K be groups. Prove that $Z(H \times K) = Z(H) \times Z(K)$.

2. Let $K \trianglelefteq G$ and $|K| = 2$. Prove that K is a central subgroup of G.

3. Prove that $K \trianglelefteq G$ implies $Z(K) \trianglelefteq G$. Show by an example that $Z(K)$ is not necessarily characteristic in G.

4. Find groups K,G such that $K \trianglelefteq G$ and $Z(K) \ntrianglelefteq Z(G)$.

5. Let G be a group and $x,y \in G$. Prove that, if $xy \in Z(G)$, then $xy = yx$.

6. Find the centers of D_4, D_{2n} (n odd), $SL(2,\mathbb{Q})$,$SL(2,\mathbb{Z})$.

7. Prove that $Z(S_n) = 1$ for $n \geqslant 3$ and $Z(A_n) = 1$ for $n \geqslant 4$.

8. Define a subgroup M of G by $M/Z(G) = Z(G/Z(G))$. Show that M is characteristic in G.

9. Let $\sigma \in Aut(G)$ and $H \leqslant G$. Prove that $H\sigma$ is a subgroup of G and is isomorphic to H.

10. Let $\varnothing \neq A \subseteq Aut(G)$ and $K \leqslant H \leqslant G$. Suppose that K is characteristic in H and $H\alpha = H$ for all $\alpha \in A$. Prove that $K\alpha = K$ for all $\alpha \in A$.

11. Show that, if $G \cong H$, then $Aut(G) \cong Aut(H)$.

12. Find all characteristic subgroups of D_8. Prove that $Inn(D_8) \neq 1$ and that $Aut(D_8) \cong D_8$.

13. Prove that $Aut(\mathbb{Z}) \cong C_2$, $Aut(V_4) \cong S_3$, $Aut(S_3) \cong S_3$, $Aut(Q_8) \cong S_4$ (see §17, Ex. 15).

14. Let H be a characteristic subgroup of G and put
$N = \{\alpha \in Aut(G)\colon (x\alpha)x^{-1} \in H \text{ for all } x \in G\}$. Prove that $N \trianglelefteq Aut(G)$.

15. Find a one-to-one homomorphism from $Aut(H \times K)$ into
$Aut(H) \times Aut(K)$.

16. Let $H \leqslant K \leqslant G$ and $\sigma \in Aut(G)$. Prove that $H\sigma \leqslant K\sigma$ and, if also $H \trianglelefteq K$, then $H\sigma \trianglelefteq K\sigma$.