§27
Series

In this paragraph, we study series of groups. The celebrated Jordan-Hölder theorem is proved and the class of solvable groups is introduced.

27.1 Definition: A nontrivial group G is called a *simple* group if G has no nontrivial proper normal subgroup.

Thus a group G is simple if and only if $G \neq 1$ and 1 and G are the only normal subgroups of G. This resembles the definition of prime numbers. Just as prime numbers are the building blocks of integers, simple groups are the building blocks of certain groups, as will be seen below. Moreover, the fundamental theorem of arithmetic has a counterpart, namely the Jordan-Hölder theorem. This theorem states that, any group G satisfying certain conditions that will be specified later, the building blocks of G are uniquely determined. However, this analogy should not be pushed too far. For one thing, the building blocks may be combined in various ways to produce different groups. Stated otherwise, different groups may have the same building blocks. In fact, the problem of determining a group from its building blocks, known as the extension problem, still awaits its solution.

It is an easy matter to find all *abelian* simple groups. Any subgroup of an abelian group is normal in that group, so an abelian group is simple if and only if it has no subgroups except 1 and itself. If G is an abelian simple group, then $G \neq 1$ by definition, and so there is an $x \in G$, $x \neq 1$. Then $\langle x \rangle$ is a nontrivial subgroup of G and, since G is simple, $\langle x \rangle$ has to be G. Thus $G = \langle x \rangle$ is cyclic. Now an infinite cyclic group has subgroups of every index (Lemma 11.11) and cannot be simple. Therefore G is a finite cyclic group, say $|G| = n > 1$. Then, for every positive divisor m of n, the group G has a subgroup of order m (Lemma 11.10). But the order of any subgroup of G is either 1 or n. Hence 1 and n are the only positive divisors of n and n is prime. Thus an abelian simple group is a cyclic group of prime order. Conversely, a cyclic group of prime order has no nontrivial proper subgroup by Lagrange's theorem, and is therefore an abelian simple group. We proved the following theorem.

27.2 Theorem: *An abelian group G is simple if and only if G is cyclic of prime order.* □

We prove next that the alternating groups A_n, where $n \geqslant 5$, are simple. We need a lemma. Let us recall that a 3-cycle is a permutation of the form (abc), $a \neq b \neq c \neq a$.

27.3 Lemma: *If $n \geqslant 3$, then A_n is generated by the set of all 3-cycles in A_n.*

Proof: We must prove that every element of A_n can be written as a product of 3-cycles (Lemma 24.2). Every element of A_n can be written as a product of an even number of transpositions and, taking the transposi-tions in pairs, we see that every element of A_n can be written as a pro-duct of permutations of the form $(ab)(cd)$, where $a \neq b$ and $c \neq d$. Hence it suffices to prove that every permutations of the form $(ab)(cd)$ can be written as a product of 3-cycles.

There are three cases to consider, in which two or one or none of c,d is in the set $\{a,b\}$. In the first case, $\{c,d\} = \{a,b\}$, hence $(cd) = (ab)$ and $(ab)(cd) = (ab)(ab) = \iota = (abe)(abe)(abe)$ is a product of 3-cycles, where e is distinct from a and b (here we use the assumption $n \geqslant 3$). In the second case, we may assume $c = a$ without loss of generality. Then $(ab)(cd) = (ab)(ad) = (abd)$ is a product of one 3-cycle. In the third case, a,b,c,d are all distinct and $(ab)(cd) = (abc)(adc)$ is a product of two 3-cycles. The proof is complete. □

27.4 Theorem: *If $n \geqslant 5$, then A_n is simple.*

Proof: Let $1 \leqslant N \lhd A_n$. We will prove $N = 1$.

First we prove that there can be no 3-cycle in N. Assume, by way of contradiction, that there is a 3-cycle (abc) in N. Let $(a'b'c')$ be any 3-cycle and choose two distinct numbers e,f from $\{1,2,\dots,n\}\backslash\{a',b',c'\}$. This is possible because $n \geqslant 5$. Let σ be a permutation in S_n such that $a\sigma = a'$, $b\sigma = b'$ and $c\sigma = c'$ and put $\pi = \sigma(ef)$. Then $a\pi = a'$, $b\pi = b'$ and $c\pi = c'$ as well and $\sigma^{-1}(abc)\sigma = (a'b'c') = \pi^{-1}(abc)\pi$. Since the signs of σ and $\pi = \sigma(ef)$ are different, either σ or π is in A_n. Then, since $(abc) \in N$ and $N \lhd A_n$, either $\sigma^{-1}(abc)\sigma$ or $\pi^{-1}(abc)\pi$ is in N. So $(a'b'c') \in N$ and N contains *all* 3-cycles. From Lemma 27.3, we conclude $A_n \leqslant N$, contrary to $N \lhd A_n$. Therefore there can be no 3-cycle in N.

Secondly, there can be no permutation in N involving a cycle of length greater than or equal to 4 when written out as a product of disjoint cycles. Indeed, if $\sigma = (abcd\dots)\pi \in N$, where $(abcd\dots)$ and π are disjoint permutations, then

$$\begin{aligned}
\sigma^{-1}.(abc)^{-1}\sigma(abc) &= \pi^{-1}(\dots dcba)(cba)(abcd\dots)\pi(abc)\\
&= \pi^{-1}\pi(\dots dcba)(cba)(abcd\dots)(abc)\\
&= (\dots dcba)(cba)(abcd\dots)(abc)\\
&= (abd)
\end{aligned}$$

would be in N, contrary to what we proved above. So the disjoint cycles of a nonidentity permutation in N have lengths (1, in which case we do not write them, or) 2 or 3.

Thirdly, in the disjoint cycle decomposition of any nonidentity element of N, there can be no 3-cycle. To prove this, we first note that, if there were only *one* 3-cycle in the disjoint cycle decomposition of a nonidentity element of N, so that its disjoint cycle decomposition is a 3-cycle times a product of transpositions, then the square of that element would be a 3-cycle in N, which is impossible. Thus, if there is a σ in N whose disjoint cycle decomposition involves a 3-cycle at all, then there are at least *two* 3-cycles in the disjoint cycle decomposition of σ. Then we have $\sigma = (abc)(def)\pi$, say, where $(abc),(def),\pi$ are disjoint permutations and

$$\begin{aligned}
\sigma.(dec)^{-1}\sigma(dec) &= (abc)(def)\pi(ced)(abc)(def)\pi(dec)\\
&= (abc)(def)(ced)(abc)(def)(dec)\pi^2\\
&= (adcbf)\pi^2
\end{aligned}$$

is in N, which is impossible, since there is a cycle of length 5 in its disjoint cycle decomposition $\big((adcbf)$ and π^2 are disjoint permutations$\big)$.

Hence, in the disjoint cycle decomposition of any nonidentity element of N, there is no cycle of length 3. Combining this with what we proved above, we conclude that any nonidentity element in N must be a product of (an even number of) disjoint transpositions.

Fourthly, a product of $2k$ disjoint transpositions cannot belong to N if k is greater than or equal to 2, for if $\sigma = (ab)(cd)(ef)(gh)\pi$ belonged to N, where $\pi = \iota$ or π is a product of disjoint transpositions and disjoint from $(ab)(cd)(ef)(gh)$, then $\sigma.(de)^{-1}(bc)^{-1}\sigma(bc)(de)$

$$= (ab)(cd)(ef)(gh)\pi(ed)(cb)(ab)(cd)(ef)(gh)\pi(bc)(de)$$
$$= (aed)(bcf)\pi^2$$

would also belong to N. This possibility was excluded above.

Since we assume $N \neq 1$, there is a $\sigma \in N$, $\sigma \neq \iota$. Here σ is necessarily a product of two disjoint transpositions, say $\sigma = (ab)(cd)$. We choose a number e from $\{1,2, \ldots ,n\}\backslash\{a,b,c,d\}$. Then the 3-cycle $\sigma.(aeb)^{-1}\sigma(aeb) = (ab)(cd)(bea)(ab)(cd)(aeb) = (abe)$ belongs to N as well, the final contradiction. This shows that the assumption $N \neq 1$ is untenable. Thus $N = 1$ and A_n is simple. □

27.5 Definition: Let G be a nontrivial group. A proper normal subgroup M of G is said to be a *maximal normal subgroup* of G if there is no subgroup L of G such that $M < L \lhd G$. Equivalently, M is a maximal normal subgroup of G if $M \lhd G$, and $M \leqslant K \trianglelefteq G$ implies that either $M = K$ or $K = G$.

27.6 Lemma: *Let $M \lhd G$. Then G/M is a simple group if and only if M is a maximal normal subgroup of G.*

Proof: Since $M \lhd G$, we have $G/M \neq 1$. If G/M is not simple, there is a normal subgroup of G/M, say N/M, which is distinct from M/M and G/M, so $M/M < N/M \lhd G/M$. By Theorem 21.2, $M < N \lhd G$ and M is not a maximal normal subgroup of G. Conversely, if M is not a maximal normal subgroup of G, there is an N such that $M < N \lhd G$ and, by Theorem 21.2, $M/M < N/M \lhd G/M$. So G/M has a nontrivial proper normal subgroup N/M and G/M is not simple. □

27.7 Definitions: Let $H \leqslant G$. A finite sequence of subgroups of G, including H and G, is called a *series from H to G*, or a *series between H and G*, if each group in the sequence is a normal subgroup of the next one. Thus a series from H to G can be written

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{n-1} \trianglelefteq H_n = G.$$

(1)

The subgroups $H_0, H_1, \ldots, H_{n-1}, H_n$ are called the *terms* of the series (1). The factor groups $H_1/H_0, H_2/H_1, \ldots, H_n/H_{n-1}$ are called the *factors* of the series (1). A series from 1 to G will be called shortly a *series of G*.

If each term $H_0, H_1, \ldots, H_{n-1}, H_n$ of the series (1) happens to be normal (characteristic) in G, the series (1) will be called a *normal* (*characteristic*) *series*.

There may be repetitions in (1). If, however, $H_{i-1} \triangleleft H_i$ for each $i = 1,2,\ldots,n$, the series (1) will be called a *proper series*.

A series

$$H = J_0 \trianglelefteq J_1 \trianglelefteq \cdots \trianglelefteq J_{m-1} \trianglelefteq J_m = G.$$

(2)

from H to G is said to be a *refinement of* (1) if every term of (1) is also a term of (2). Thus a refinement of (1) is obtained from (1) by inserting additional groups between some consecutive terms of (1). These additional terms need not be distinct from the terms of (1). For example, $A \trianglelefteq B \trianglelefteq B \trianglelefteq C$ is a refinement of $A \trianglelefteq B \trianglelefteq C$. If (2) is a refinement of (1) and if there is at least one term in (2) which is not a term of (1), then (2) is called a *proper refinement of* (1).

27.8 Definition: Let G be a group. A series of G is called a *composition series of G* if it is a proper series of G and has no proper refinement. A factor of a composition series of G is called a *composition factor of G*.

27.9 Lemma: *A series*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

of a group G is a composition series of G if and only if all factors G_i/G_{i-1} ($i = 1,2,\ldots,n$) are simple.

Proof: Suppose first that the given series is a composition series of G. By definition, it is a proper series. So $G_{i-1} \lhd G_i$ and all factors G_i/G_{i-1} are distinct from the trivial group ($i = 1,2,\ldots,n$). If one of the factors, say G_j/G_{j-1}, were not simple, G_j/G_{j-1} would have a nontrivial proper normal subgroup, which may be written as H/G_{j-1}, where $G_{j-1} < H < G_j$ by Theorem 21.2. Hence $G_{j-1} \lhd H \lhd G_j$ (in fact $G_{j-1} \lhd G_j$) and the given series has a proper refinement which is obtained by inserting H between G_{j-1} and G_j, contrary to our hypothesis that the given series is a composition series. Hence G_i/G_{i-1} are all simple ($i = 1,2,\ldots,n$).

Conversely, let us assume that all factors G_i/G_{i-1} are simple ($i = 1,2,\ldots,n$). Then G_i/G_{i-1} is not trivial and so $G_{i-1} \lhd G_i$ for all $i = 1,2,\ldots,n$. Thus the given series is proper. If it were not a composition series, it would have a proper refinement. To fix the ideas, let us assume that such a refinement has a term H between G_j and G_{j-1}, so that $G_{j-1} \lhd H \lhd G_j$. By Theorem 21.2, H/G_{j-1} would be a nontrivial proper normal subgroup of G_j/G_{j-1}, contrary to the hypothesis that all factors, including G_j/G_{j-1}, are simple. Hence the given series is a composition series. □

27.10 Examples: (a) $1 \lhd S_3$ is a series of S_3 and $1 \lhd A_3 \lhd S_3$ is a refinement thereof. The latter is a composition series of S_3, because the factors $A_3/1 \cong C_3$ and $S_3/A_3 \cong C_2$ are simple (Theorem 27.2, Lemma 27.9). It is easily seen that $1 \lhd A_3 \lhd S_3$ is the unique composition series of S_3 (cf. §15, Ex.10).

(b) $1 \lhd V_4 \lhd A_4 \lhd S_4$ is a normal series of S_4 (it is a chief series of S_4; see Ex. 4). It is not a composition series of S_4, for it can be refined by inserting one of the subgroups $U_1 = \{\iota,(12)(34)\}$, $U_2 = \{\iota,(13)(24)\}$, and $U_3 = \{\iota,(14)(23)\}$ between 1 and $V_4 = \{\iota,(12)(34),(13)(24),(14)(23)\}$. Each one of the three series $1 \lhd U_i \lhd V_4 \lhd A_4 \lhd S_4$ is a composition series of S_4 ($i = 1,2,3$). The reader will easily verify that these are the only composition series of S_4.

(c) We want to find all composition series of S_n for $n \geqslant 5$. For this purpose, we determine all normal subgroups of S_n.

Let $n \geqslant 5$ and $1 < N \trianglelefteq S_n$. Then $N \cap A_n \trianglelefteq A_n$ by Theorem 21.3 and, since A_n is simple (Theorem 27.4), either $N \cap A_n = A_n$ or $N \cap A_n = 1$.

In case $N \cap A_n = A_n$, we have $A_n \leqslant N \leqslant S_n$. Thus $|N{:}A_n|$ divides $|S_n{:}A_n| = 2$ and $|N{:}A_n| = 1$ or $|N{:}A_n| = 2$. Hence $N = A_n$ or $N = S_n$.

In case $N \cap A_n = 1$, we have $N \nleqslant A_n$ (because $1 < N$), so $A_n < A_nN \leqslant S_n$, so $A_nN = S_n$ and $|N| = |N{:}1| = |N{:}N \cap A_n| = |A_nN{:}A_n| = |S_n{:}A_n| = 2$. Thus $N = \{\iota,\sigma\}$ for some $\sigma \in S_n \backslash A_n$. Since $N \trianglelefteq S_n$, we obtain

$$\{\iota,\sigma\} = N = N^\tau = \{\iota,\sigma\}^\tau = \{\iota^\tau,\sigma^\tau\} = \{\iota,\sigma^\tau\} \text{ for all } \tau \in S_n,$$

hence

$$\sigma^\tau = \sigma \qquad \text{for all } \tau \in S_n. \tag{$*$}$$

From $o(\sigma) = |\langle\sigma\rangle| = |N| = 2$, we see that the disjoint cycle decomposition of σ involves transpositions only (Theorem 15.17), say

$$\sigma = (a_1b_1)(a_2b_2)\dots(a_mb_m)$$

for some odd number m. If $m \geqslant 3$, then $\pi := (a_3b_3)\dots(a_mb_m)$ is disjoint from $(a_1b_1)(a_2b_2)$ and

$$\sigma^{(a_1a_2)} = (a_1b_1)^{(a_1a_2)}(a_2b_2)^{(a_1a_2)}\pi^{(a_1a_2)} = (a_2b_1)(a_1b_2)\pi \neq (a_1b_1)(a_2b_2)\pi = \sigma$$

contrary to $(*)$. Hence $m = 1$ and $\sigma = (a_1b_1)$. Let $c \in \{1,2,\dots,n\}\backslash\{a_1,b_1\}$. Now

$$\sigma^{(a_1c)} = (ca_1)(a_1b_1)(a_1c) = (b_1c) \neq (a_1b_1) = \sigma,$$

again contradicting $(*)$. Hence there is no nontrivial normal subgroup N of S_n such that $N \cap A_n = 1$.

Consequently, $1, A_n, S_n$ are the only normal subgroups of S_n when $n \geqslant 5$. Thus if $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{k-1} \trianglelefteq G_k = S_n$ is a composition series of S_n, here G_{k-1} has to be A_n and G_{k-2} has to be 1. Therefore the series must be $1 \triangleleft A_n \triangleleft S_n$, which is indeed a composition series of S_n, for $A_n/1 \cong A_n$ and $S_n/A_n \cong C_2$ are simple groups (Lemma 27.9).

Thus $1 \triangleleft A_n \triangleleft S_n$ is the unique composition series of S_n when $n \geqslant 5$.

(d) Not every group has a composition series. For example, $\mathbb{Z}$ has no composition series. Indeed, any series of $\mathbb{Z}$ is of the form

$$0 \lhd m_1\mathbb{Z} \unlhd m_2\mathbb{Z} \unlhd \ldots \unlhd m_n\mathbb{Z} \unlhd \mathbb{Z}, \qquad (3)$$

where $m_2|m_1$, $m_3|m_2$, ..., $m_n|m_{n-1}$. If m_0 is a multiple of m_1 and $m_0 \neq m_1$, then

$$0 \unlhd m_0\mathbb{Z} \lhd m_1\mathbb{Z} \unlhd m_2\mathbb{Z} \unlhd \ldots \unlhd m_n\mathbb{Z} \lhd \mathbb{Z}$$

is a proper refinement of (3). Thus any series of $\mathbb{Z}$ has a proper refinement. Consequently, no series of $\mathbb{Z}$ can be a composition series of $\mathbb{Z}$.

(e) Let $<a>$ be a cyclic group of order 12. Then
$1 \lhd <a^6> \lhd <a^2> \lhd <a>$; $1 \lhd <a^6> \lhd <a^3> \lhd <a>$; $1 \lhd <a^4> \lhd <a^2> \lhd$
$<a>$
are the composition series of $<a>$. The composition factors are isomorphic to C_2, C_3, C_2; C_2, C_2, C_3; C_3, C_2, C_2.
Thus, aside from order, the composition factors arising from different composition series are isomorphic groups.

27.11 Definition: Let G be a group. Two series

$$1 = G_0 \unlhd G_1 \unlhd \cdots \unlhd G_{n-1} \unlhd G_n = G$$
$$1 = H_0 \unlhd H_1 \unlhd \cdots \unlhd H_{m-1} \unlhd H_m = G$$

of G are said to be *equivalent* if $n = m$ and if the factors G_i/G_{i-1} are, in some order, isomorphic to the factors H_j/H_{j-1} $(i,j = 1,2, \ldots, n)$.

Here it is not stipulated that $G_i/G_{i-1} \cong H_i/H_{i-1}$ for all $i = 1,2, \ldots, n$. The condition in Definition 27.11 is that $G_i/G_{i-1} \cong H_{i\sigma}/H_{i\sigma-1}$ for some $\sigma \in S_n$. Clearly, Definition 27.11 introduces an equivalence relation on the set of all series of G. The three series in Example 27.10(e) are equivalent. We will prove that any two composition series of a group are equivalent, provided G does have a composition series (Jordan-Hölder theorem). In fact, a much stronger theorem is true (see Schreier's theorem below). We need some elementary results.

27.12 Lemma (Dedekind's modular law): *Let G be a group and let A,B,C be subgroups of G such that $A \leqslant C$. Then*

$$A(B \cap C) = AB \cap C.$$

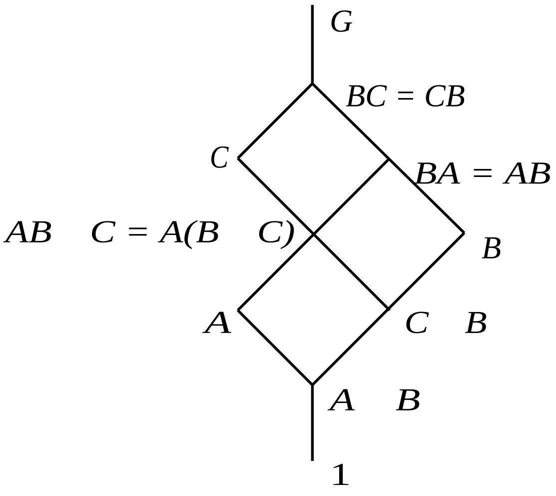
($A(B \cap C)$ and $AB \cap C$ are not necessarily subgroups of G.)

Proof: Let $x \in A(B \cap C)$. Then $x = ab$ for some $a \in A$, $b \in B \cap C$. Thus $x = ab \in AB$ and $x = ab \in AC = C$, so $x \in AB \cap C$. This gives $A(B \cap C) \subseteq AB \cap C$. To show the reverse inclusion, let $c \in AB \cap C$. Then $c = a_1b_1$ for some a_1 in A and b_1 in B. From $b_1 = a_1^{-1}c \in AC = C$, we conclude $b_1 \in B \cap C$, hence $c = a_1b_1 \in A(B \cap C)$. This gives $AB \cap C \subseteq A(B \cap C)$. So $A(B \cap C) = AB \cap C$. □

27.13 Lemma: *Let $A \trianglelefteq C \leqslant G$ and $B \leqslant G$. Then*

$$A \cap B \trianglelefteq C \cap B \quad \textit{and} \; C \cap B \,/\, A \cap B \cong A(B \cap C)/A.$$

Proof: If G is a group, $H \trianglelefteq G$ and $K \leqslant G$, then $H \cap K \trianglelefteq K$ and $K \,/\, H \cap K$ is isomorphic to HK/H by Theorem 21.3. Using this theorem with G,H,K replaced by $C,A,C \cap B$, respectively, we obtain $A \cap (C \cap B) \trianglelefteq C \cap B$ and $C \cap B \,/\, A \cap (C \cap B) \cong A(C \cap B)/A$. Since $A \cap (C \cap B) = A \cap B$ and $A(C \cap B) = A(B \cap C)$, the claim follows. □



27.14 Lemma: *Let $A \trianglelefteq C \leqslant G$ and $B \trianglelefteq G$. Then*

$$BA \trianglelefteq BC \quad \textit{and} \quad BC/BA \cong C/A(B \cap C).$$

Proof: Since $B \trianglelefteq G$, we know from Lemma 19.4 that $AB = BA \leqslant G$ and that $CB = BC \leqslant G$. Thus $BA \leqslant BC$. We prove next that BA is normal in BC. We observe

$$B \leqslant BA \leqslant N_G(BA)$$

hence $$(BA)^b = BA \qquad \text{for all } b \in B.$$

Then, for any $b \in B$, $c \in C$, we obtain

$$(BA)^{bc} = [(BA)^b]^c = (BA)^c = B^cA^c = BA^c = BA$$

since $B \trianglelefteq G$ and $A \trianglelefteq C$. Thus $(BA)^x = BA$ for all $x \in BC$ and $BA \trianglelefteq BC$.

Using Theorem 21.3 with BC, BA, C in place of G,H,K, respectively, we get

$$AB \cap C \trianglelefteq C \text{ and } C/AB \cap C \cong C(AB)/AB.$$

Since $AB \cap C = A(B \cap C)$ and $C(AB) = (CA)B = CB = BC$, this isomorphism means $C/A(B \cap C) \cong BC/BA$. □

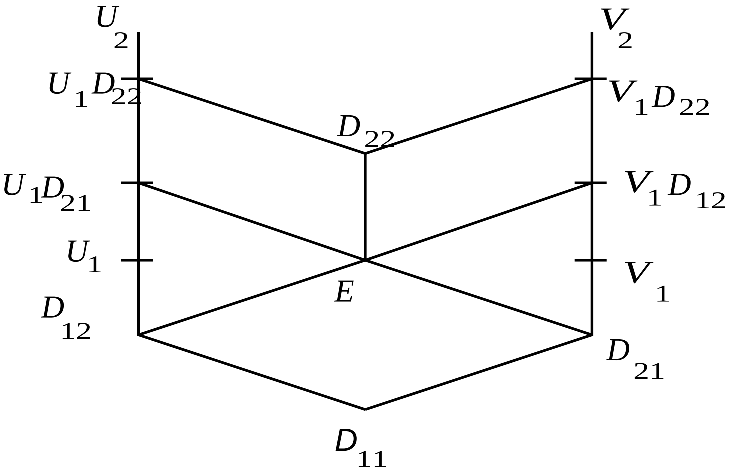
27.15 Lemma (Zassenhaus' lemma): *Let G be a group,*

$$U_1 \trianglelefteq U_2 \leqslant G \text{ and } V_1 \trianglelefteq V_2 \leqslant G.$$

Then $U_1(U_2 \cap V_1) \trianglelefteq U_1(U_2 \cap V_2)$, $\qquad V_1(U_1 \cap V_2) \trianglelefteq V_1(U_2 \cap V_2)$ *and*

$$U_1(U_2 \cap V_2)\,/U_1(U_2 \cap V_1) \cong V_1(U_2 \cap V_2)\,/\,V_1(U_1 \cap V_2).$$

Proof: We put $D_{ij} := U_i \cap V_j$ $(i,j = 1,2)$. Since $U_1 \trianglelefteq U_2$, we have $U_1 \cap V_2 \trianglelefteq U_2 \cap V_2$ by Lemma 27.13, so $D_{12} \trianglelefteq D_{22}$. Similarly,$V_1 \trianglelefteq V_2$ and Lemma 27.13 gives $U_2 \cap V_1 \trianglelefteq U_2 \cap V_2$, so $D_{21} \trianglelefteq D_{22}$. Now $D_{12} \trianglelefteq D_{22}$ and $D_{21} \trianglelefteq D_{22}$ and writing $E = D_{12}D_{21} = D_{21}D_{12}$ for brevity, we get $E \trianglelefteq D_{22}$ by Lemma 19.4(3).



Since $E \trianglelefteq D_{22} \leqslant U_2$ and $U_1 \trianglelefteq U_2$, Lemma 27.14 gives

$$U_1E \trianglelefteq U_1D_{22} \quad\text{and}\quad U_1D_{22}/U_1E \cong D_{22}/E(U_1 \cap D_{22}). \qquad (4)$$

Here $U_1E = U_1(D_{12}D_{21}) = (U_1D_{12})D_{21} = U_1D_{21}$ and $E(U_1 \cap D_{22}) = E$ (because $U_1 \cap D_{22} = D_{12} \subseteq E$), so (4) becomes

$$U_1D_{21} \trianglelefteq U_1D_{22} \quad \text{and} \quad U_1D_{22}/U_1D_{21} \cong D_{22}/E. \tag{5}$$

Repeating the same argument with U's replaced by V's, we get

$$V_1D_{12} \trianglelefteq V_1D_{22} \quad \text{and} \quad V_1D_{22}/V_1D_{12} \cong D_{22}/E. \tag{6}$$

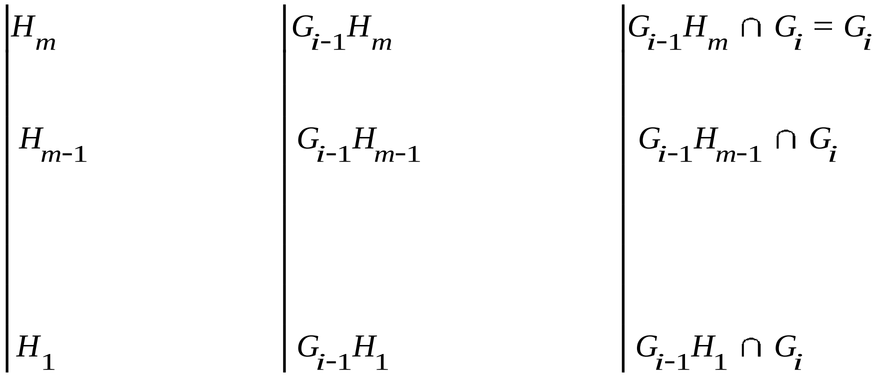
The claim follows from (5) and (6). □

27.16 Theorem (Schreier): *Any two series of a group have equivalent refinements. More precisely, if*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G \tag{g}$$
$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = G \tag{h}$$

are series of G, then there are series (g') and (h') of G such that (g') is a refinement of (g), (h') is a refinement of (h), and (g') and (h') are equivalent.

Proof: We will try to build a series between each G_{i-1} and G_i ($i = 1,2,\dots,n$) by so modifying the terms of (h) that the modified series begin from G_{i-1} and terminate at G_i. There are two natural ways of doing this. Either we multiply each term of (h) by G_{i-1} (the resulting series will thus begin from G_{i-1}) and intersect the products with G_i (the modified series will thus terminate at G_i); or we intersect each term of (h) by G_i (the resulting series will thus terminate at G_i) and multiply the intersections by G_{i-1}(the modified series will thus begin from G_{i-1}). By Dedekind's modular law (Lemma 27.12), these two series between G_{i-1} and G_i are identical.



H_0	$G_{i-1}H_0$	$G_{i-1}H_0 \cap G_i = G_{i-1}$

H_m	$H_m \cap G_i$	$G_{i-1}(H_m \cap G_i) = G_i$
H_{m-1}	$H_{m-1} \cap G_i$	$G_{i-1}(H_{m-1} \cap G_i)$
H_1	$H_1 \cap G_i$	$G_{i-1}(H_1 \cap G_i)$
H_0	$H_0 \cap G_i$	$G_{i-1}(H_0 \cap G_i) = G_{i-1}$

We put $G_{ij} = G_{i-1}H_j \cap G_i = G_{i-1}(H_j \cap G_i)$ $(i = 1,2, \ldots ,n;\ j = 1,2, \ldots ,m)$.

Similarly, we put $H_{ij} = H_{j-1}G_i \cap H_j = H_{j-1}(G_i \cap H_j)$ $(i = 1,2, \ldots ,n;\ j = 1,2, \ldots ,m)$.

Here $G_{i-1} \trianglelefteq G_i$, hence $G_{i-1}(H_j \cap G_i) \leqslant G_i$ by Lemma 19.4(2). Thus G_{ij} is a subgroup of G_i. In the same way, H_{ij} is a subgroup of H_i. So

$$G_{i-1} = G_{i0} \leqslant G_{i1} \leqslant G_{i2} \leqslant \cdots \leqslant G_{i,m-1} \leqslant G_{im} = G_i \qquad (g_i)$$

and
$$H_{j-1} = H_{0j} \leqslant H_{1j} \leqslant H_{2j} \leqslant \cdots \leqslant H_{n-1,j} \leqslant H_{nj} = H_j. \qquad (h_j)$$

Using Zassenhaus' lemma (Lemma 27.15) with

$$U_1 = G_{i-1},\ U_2 = G_i,\ V_1 = H_{j-1},\ V_2 = H_j,$$

we obtain, for each $i = 1,2, \ldots ,n,\ j = 1,2, \ldots ,m$:

$$G_{i-1}(G_i \cap H_{j-1}) \trianglelefteq G_{i-1}(G_i \cap H_j),\ \ H_{j-1}(G_{i-1} \cap H_j) \trianglelefteq H_{j-1}(G_i \cap H_j)$$

and
$$G_{i-1}(G_i \cap H_j) \,/\, G_{i-1}(G_i \cap H_{j-1}) \cong H_{j-1}(G_i \cap H_j) \,/\, H_{j-1}(G_{i-1} \cap H_j).$$

Thus $G_{i,j-1} \trianglelefteq G_{ij}$, $H_{i-1,j} \trianglelefteq H_{ij}$ and $G_{ij}/G_{i,j-1} \cong H_{ij}/H_{i-1,j}$.

Therefore (g_i) is a series between G_{i-1} and G_i, and (h_j) is a series between H_{j-1} and H_j. Writing the terms of $(g_1),(g_2),(g_3), \ldots ,(g_n)$ consecutively, we obtain a series (g') of G with nm factors; and writing the terms of $(h_1),(h_2),(h_3), \ldots ,(h_m)$ consecutively, we obtain a series (h') of H with mn

factors. Here (g') is a refinement of (g) and (h') is a refinement of (h). Finally, in view of the isomorphisms $G_{ij}\,/G_{i,j-1} \cong H_{ij}/H_{i-1,j}$, the series ($g'$) and ($h'$) are equivalent. □

27.17 Theorem: *Let G be a group and assume that G has a composition series.*

(1) *Every proper series of G has a refinement which is a composition series.*

(2) (Jordan-Hölder Theorem) *Any two composition series of G are equivalent.*

Proof: Let

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G \qquad (g)$$

be a proper series of G and let

$$1 = H_0 \lhd H_1 \lhd \cdots \lhd H_{m-1} \lhd H_m = G \qquad (h)$$

be a composition series of G. By Schreier's theorem (Theorem 27.16), there are equivelent series (g') and (h') of G such that (g') is a refinement of (g) and (h') is a refinement of (h). From (g') and (h'), we delete repeated factors and thereby obtain two equivalent proper series, say (g'') and (h''), respectively. Here (g'') is a refinement of (g) and (h'') is a refinement of (h), because both (g) and (h) are proper series.

(1) (h'') is a proper series and is a refinement of (h). But (h) has no proper refinement, because (h) is a composition series. Hence (h'') is identical with (h). Thus (g) has a refinement (g'') which is equivalent to the composition series (h'') = (h). Then the factors of (g''), being isomorphic to the composition factors in (h), are all simple groups and (g'') itself is a composition series by Lemma 27.9. Therefore any proper series (g) of G has a refinement (g'') which is a composition series.

(2) Assume now (g) is also a composition series of G. By the same argument as above, (g'') must be identical with (g). Then (g) = (g'') and (h'') = (h) are equivalent. Thus any two composition series of G are equivalent. □

We now discuss the class of solvable groups.

27.18 Definition: A series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = G$$

from H to G is said to be an *abelian* series if all the factors

$$H_1/H_0, H_2/H_1, \ldots, H_m/H_{m-1}$$

are abelian groups.

27.19 Definition: A group G is called a *solvable* (or *soluble*) group if G has an abelian series (from 1 to G).

Clearly, any abelian group A is solvable: $1 \trianglelefteq A$ is an abelian series of A. S_3 is an example of a nonabelian solvable group. Not every group is solvable. For example nonabelian simple groups are certainly not solvable. In particular, A_n is not solvable for $n \geqslant 5$.

27.20 Lemma: *If G is a solvable group, then all subgroups and factor groups of G are solvable.*

Proof: Being solvable, G has an abelian series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = G.$$

Let K be an arbitrary subgroup of G. Then, by Lemma 27.13,

$$1 = H_0 \cap K \trianglelefteq H_1 \cap K \trianglelefteq \cdots \trianglelefteq H_{m-1} \cap K \trianglelefteq H_m \cap K = G \cap K = K \qquad (6)$$

is a series of K and $H_i \cap K \,/\, H_{i-1} \cap K \cong H_{i-1}(K \cap H_i)/H_i \leqslant H_i/H_{i-1}$ for all $i = 1,2,\ldots,m$. Since H_i/H_{i-1} are abelian, $H_i \cap K \,/\, H_{i-1} \cap K$ are also abelian and (6) is an abelian series of K. Hence K is solvable.

Now let N be an arbitrary normal subgroup of G. By Lemma 27.14 and Theorem 21.2,

$$N/N = H_0N/N \trianglelefteq H_1N/N \trianglelefteq \cdots \trianglelefteq H_{m-1}N/N \trianglelefteq H_mN/N = G/N \qquad (7)$$

is a series of G/N, and, for all $i = 1,2, \ldots ,m$, $H_iN/N \big/ H_{i-1}N/N \cong H_iN/H_{i-1}N \cong H_i/H_{i-1}(N \cap H_i) \cong H_i/H_{i-1} \big/ H_{i-1}(N \cap H_i)/H_{i-1}$ is a factor group of the abelian group H_i/H_{i-1} and therefore $H_iN/N \big/ H_{i-1}N/N$ is abelian (Lemma 18.9(2)). So (7) is an abelian series of G/N and G/N is solvable. □

27.21 Lemma: *Let $N \trianglelefteq G$. If N and G/N are both solvable, then G is solvable.*

Proof: By hypothesis, there are an abelian series

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{m-1} \trianglelefteq N_m = N$$

of N and an abelian series

$$N/N = H_0/N \trianglelefteq H_1/N \trianglelefteq \cdots \trianglelefteq H_{k-1}/N \trianglelefteq H_k/N = G/N$$

of G/N. By Theorem 21.2,

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{m-1} \trianglelefteq N_m = N = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = G$$

is a series of G. Since N_j/N_{j-1} is abelian for $j = 1,2, \ldots ,m$ and $H_i/H_{i-1} \cong H_i/N \big/ H_{i-1}/N$ is abelian for $i = 1,2, \ldots ,k$, this is an abelian series of G. Thus G is solvable. □

27.22 Lemma: *Let H and K be normal solvable subgroups of a group G. Then HK is a normal solvable subgroup of G.*

Proof: HK is a normal subgroup of G by Lemma 19.4(3). Also, since K is solvable, $K/H \cap K$ is solvable by Lemma 27.20, so HK/H is solvable by Theorem 21.3. So H and HK/H are solvable and consequently HK is solvable by Lemma 27.21. □

27.23 Theorem: *If G is a finite p-group, then G is solvable.*

Proof: If G is a finite p-group of order $|G| = p^a$, then there is a series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{a-1} \trianglelefteq H_a = G$$

of G whose factors H_i/H_{i-1} $(i = 1,2, \dots ,a)$ are cyclic of order p (Theorem 26.3(2)). Thus G has an abelian series and G is solvable. □

The series in Theorem 27.23 is a composition series of G. We now want to prove more generally that a finite group G is solvable if and only if every composition factor of G is cyclic of prime order. A finite group does have a composition series, of course.

27.24 Lemma: *A solvable group G is simple if and only if G is cyclic of prime order.*

Proof: Let G be a simple solvable group. Then $G \neq 1$ and G has an abelian series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = G.$$

After deleting repetitions, we may assume that this is a proper series. Then $H_{m-1} \lhd G$ and G/H_{m-1} is a nontrivial abelian group. Thus $G' \leqslant H_{m-1}$ (Theorem 24.14) and G' is a proper normal subgroup of G. Since G is simple, $G' = 1$ and G is abelian. Thus G is cyclic of prime order by Theorem 27.2. Conversely, a cyclic group of prime order is simple; and abelian, hence solvable. □

27.25 Theorem: *Let G be a finite group. G is solvable if and only if every composition factor of G has prime order.*

Proof: If G is solvable, then any composition factor of G is solvable by Lemma 27.20, simple by Lemma 27.9, and so has prime order by Lemma 27.24. Conversely, if every composition factor of G has prime order, then a composition series of G is an abelian series of G and therefore G is solv-able.

□

The following result will play a crucial role in proving that a polynomial equation of degree greater than four cannot be solved by radicals.

27.26 Theorem: *If $n \geqslant 5$, then S_n is not solvable.*

Proof: Otherwise the subgroup A_n of S_n would be solvable (Lemma 27.20), whereas A_n, being a nonabelian simple group (Lemma 27.3), cannot have an abelian series. The conclusion follows also from Theorem 27.25, since A_n is a composition factor of S_n $(1 \lhd A_n \lhd S_n$ is the unique composition series of S_n by Example 27.10(c)$)$. □

Exercises

1. Let $\{G_i : i \in \mathbb{N}\}$ be a collection of simple groups such that $G_i \leqslant G_{i+1}$ for all $i \in \mathbb{N}$ and let $G = \bigcup_{i=1}^{\infty} G_i$. Prove that G is a simple group.

2. Let $S_{(\mathbb{N})} = \{\sigma \in S_{\mathbb{N}} : k\sigma \neq k$ for at most finitely many $k \in \mathbb{N}\}$ and, for each $n \in \mathbb{N}$, let $S(n) = \{\sigma \in S_{\mathbb{N}} : k\sigma \neq k$ for all $k \geqslant n + 1\}$. Show that $S_n \cong S(n) \leqslant S_{(\mathbb{N})} \leqslant S_{\mathbb{N}}$. Let $A(n)$ denote the image of A_n under the isomorphism $S_n \cong S(n)$ for $n \geqslant 2$ and show that $A := \bigcup_{i=5}^{\infty} A(n)$ is a simple group. (A is called the *infinite alternating group of degree* $|\mathbb{N}|$.)

3. Let $M \lhd G$ and $|G{:}M|$ be prime. Prove that M is a maximal normal subgroup of G.

4. A normal series of a group G is called a *chief series of G* if it is a proper series and if it has no proper refinement which is a normal series of G. A factor of a chief series of G is called a *chief factor of G*.

Let G be a nontrivial group. A nontrivial normal subgroup M of G is called a *minimal normal subgroup of G* if there is no $L \trianglelefteq G$ such that $1 < L < M$.

Prove the following statements.

(a) H/K is a chief factor of G if and only if H/K is a minimal subgroup of G/K.

(b) If M is a minimal normal subgroup of G, then M has no characteristic subgroup except 1 and M.

(c) If G has a composition series, then G has a chief series.

(d) $1 \lhd V_4 \lhd A_4 \lhd S_4$ is the unique chief series of S_4.

5. Suppose G has a finite abelian group having no characteristic subgroups except 1 and G. Show that there is a prime number p such that $g^p = 1$ for all $g \in G$.

6. Prove that an abelian group has a composition series if and only if it is finite.

7. Find an infinite abelian subgroup of the infinite alternating group (see Ex. 2). Conclude that a subgroup of a group with a composition series does not necessarily have a composition series.

8. Let $H \trianglelefteq G$. Prove that, if G has a composition series, so does G/H.

9. Let $H \trianglelefteq G$. Prove that, if H and G/H have composition series, so does G.

10. Repeat the proof of Schreier's theorem for the two series $1 \lhd C_{18} \lhd C_{36}$ and $1 \lhd C_4 \lhd C_{12} \lhd C_{36}$ of the cyclic group C_{36}.

11. Prove that, if H and K are solvable, so is $H \times K$.

12. Prove that, if $H,K \trianglelefteq G$ and G/H, G/K are solvable, so is $G/H \cap K$.

13. For each $n \in \mathbb{N}$, we define a subgroup $G^{(n)}$ of G recursively by $G^{(n+1)} = (G^{(n)})' = [G^{(n)},G^{(n)}]$. The series

$$G \geqslant G^{(1)} \geqslant G^{(2)} \geqslant \cdots$$

is called the *derived series of* G. Show that each $G^{(n)}$ is characteristic in G and that, if

$$G_r \trianglelefteq G_{r\text{-}1} \trianglelefteq G_{r\text{-}2} \trianglelefteq \ldots \trianglelefteq G_1 \trianglelefteq G_0 = G$$

is an abelian series between G_r and G, then $G^{(n)} \leqslant G_n$ for each $n = 1,2, \ldots ,r$. Prove that G is solvable if and only if $G^{(r)} = 1$ for some $r \in \mathbb{N}$.

14. Prove that a solvable group has a composition series if and only if it is finite (cf. Ex. 6).