

## §36 Fields of Rational Functions

The reader might have missed the familiar quotient rule  $(\frac{f}{g})' = \frac{fg' - fg'}{g^2}$  in Lemma 35.15. It was missing because  $\frac{f}{g}$  is not a polynomial.

We now introduce these quotients  $\frac{f}{g}$ .

**36.1 Definition:** Let  $D$  be an integral domain and  $x, x_1, x_2, \dots, x_n$  indeterminates over  $D$ . Then  $D[x]$  and  $D[x_1, x_2, \dots, x_n]$  are integral domains (Lemma 33.6, Lemma 33.10). An element in the the field of fractions of  $D[x]$  is called a *rational function (in  $x$ ) over  $D$* . The field of fractions of  $D[x]$  will be called the *field of rational functions over  $D$  (in  $x$ )* and will be denoted by  $D(x)$ . An element in the the field of fractions of  $D[x_1, x_2, \dots, x_n]$  is called a *rational function (in  $x_1, x_2, \dots, x_n$ ) over  $D$* . The field of fractions of  $D[x_1, x_2, \dots, x_n]$  will be called the *field of rational functions over  $D$  (in  $x_1, x_2, \dots, x_n$ )* and will be denoted by  $D(x_1, x_2, \dots, x_n)$ .

Thus a rational function over  $D$  is a fraction  $\frac{f}{g}$  of two polynomials over  $D$ ,

with  $g \neq 0$ . Two rational functions  $\frac{f_1}{g_1}$  and  $\frac{f_2}{g_2}$  are equal if and only if the

polynomials  $f_1g_2$  and  $g_1f_2$  are equal. Two rational functions  $\frac{f_1}{g_1}$  and  $\frac{f_2}{g_2}$  are

added and multiplied according to the rules

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1g_2 + g_1f_2}{g_1g_2}, \quad \frac{f_1}{g_1} \frac{f_2}{g_2} = \frac{f_1f_2}{g_1g_2}.$$

Here  $g_1$  and  $g_2$  are distinct from the zero polynomial over  $D$ .

This terminology is unfortunate and misleading, because a rational function is *not* a function in the sense of Definition 3.1. A rational function is *not* a function of the 'rational' kind, whatever that might mean. The technical term we defined is *rational function*, a term consisting of two words "rational" and "function". The meaning of the

words "rational" and "function" do not play any role in Definition 36.1. A rational function is a fraction of polynomials over  $D$ . The reader should exercise caution about this point. One should not conclude that

$$\frac{x^2 - 1}{x - 1} \quad \text{and} \quad \frac{x + 1}{1} \quad \text{in } \mathbb{C}(x)$$

are different rational functions, on grounds that their domains are different, since the domain of the first one does not contain 1, whereas 1 is in the domain of the second one. Neither of them has a domain, for neither of them is a function. And these rational functions are equal because the polynomials  $(x^2 - 1)1$  and  $(x - 1)(x + 1)$  in  $\mathbb{C}[x]$  are equal.

**36.2 Lemma:** *Let  $D$  be an integral domain and  $F$  the field of fractions of  $D$ . Let  $x$  be an indeterminate over  $D$ . Then  $D(x) = F(x)$ .*

**Proof:**  $F$  consists of the fractions  $\frac{a}{b}$ , where  $a, b \in D$  and  $b \neq 0$ ; and  $D(x)$  consists of the fractions

$$\frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0},$$

where  $a_n, a_{n-1}, \dots, a_1, a_0, b_m, b_{m-1}, \dots, b_1, b_0 \in D$  and the denominator is distinct from the zero polynomial in  $D[x]$ . Finally,  $F(x)$  consists of the fractions

$$\frac{c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0}{d_m x^m + d_{m-1} x^{m-1} + \cdots + d_1 x + d_0},$$

where  $c_n, c_{n-1}, \dots, c_1, c_0, d_m, d_{m-1}, \dots, d_1, d_0 \in F$  and the denominator is distinct from the zero polynomial in  $F[x]$ .

An element of  $D$  is identified with the fraction  $\frac{a}{1}$  in  $F$  (Theorem 31.5),

whence  $D \subseteq F$ . Thus  $D[x] \subseteq F[x]$  as sets. Note that two elements  $\frac{f(x)}{g(x)}$  and

$\frac{p(x)}{q(x)}$  of  $D(x)$  are equal in  $D(x)$  if and only if  $f(x)q(x) = g(x)p(x)$  in  $D[x]$ , and

this holds if and only if  $f(x)q(x) = g(x)p(x)$  in  $F[x]$ , so if and only if  $\frac{f(x)}{g(x)}$

and  $\frac{p(x)}{q(x)}$  are equal in  $F(x)$ . Thus every element of  $D(x)$  is in  $F(x)$  and equality in  $D(x)$  coincides with equality in  $F(x)$ . So  $D(x) \subseteq F(x)$ .

Next we show  $F(x) \subseteq D(x)$ . Let  $\frac{p(x)}{q(x)} \in F(x)$ , with  $p(x), q(x) \in F[x]$ ,  $q(x) \neq 0$ .

Then  $p(x) = \sum_{i=0}^n \frac{a_i}{b_i} x^i$ ,  $q(x) = \sum_{j=0}^m \frac{c_j}{d_j} x^j$ , where  $a_i, b_i, c_j, d_j \in D$ ,  $b_i \neq 0$ ,  $d_j \neq 0$  for

all  $i, j$  and not all of  $c_j$  are equal to 0. We put  $b = b_0 b_1 \dots b_{n-1} b_n$  and  $d = d_0 d_1 \dots d_{m-1} d_m$ . Then  $dbp(x)$  and  $dbq(x)$  are polynomials in  $D[x]$ , and hence

$\frac{p(x)}{q(x)} = \frac{dbp(x)}{dbq(x)} \in D(x)$ . So  $F(x) \subseteq D(x)$ . This proves  $D(x) = F(x)$ .  $\square$

As an illustration of Lemma 36.2, observe that  $\frac{\frac{2}{3}x^2 - \frac{1}{7}x + \frac{1}{4}}{\frac{2}{5}x^2 + \frac{1}{3}x - \frac{1}{2}} \in \mathbb{Q}(x)$

is equal to the rational function  $\frac{5(56x^2 - 12x + 21)}{14(12x^2 + 10x - 15)}$  in  $\mathbb{Z}[x]$ .

**36.3 Remark:** Let  $D$  be an integral domain and  $F$  the field of fractions of  $D$ . Then

$$\begin{aligned} D(x_1, x_2, \dots, x_n) &= \text{field of fractions of } D[x_1, x_2, \dots, x_n] \\ &= \text{field of fractions of } D[x_1, x_2, \dots, x_{n-1}][x_n] \\ &= D[x_1, x_2, \dots, x_{n-1}](x_n) \\ &= D(x_1, x_2, \dots, x_{n-1})(x_n) \end{aligned}$$

by Lemma 36.2, with  $D[x_1, x_2, \dots, x_{n-1}]$ ,  $D(x_1, x_2, \dots, x_{n-1})$ ,  $x_n$  in place of  $D, F, x$ , respectively.

Also, we have  $D(x_1, x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n)$ , for this is true when  $n = 1$  (Lemma 36.2) and, when it is true for  $n = k$ , so that  $D(x_1, x_2, \dots, x_k) = F(x_1, x_2, \dots, x_k)$ , it is also true for  $n = k + 1$ :

$$\begin{aligned} D(x_1, x_2, \dots, x_k, x_{k+1}) &= D(x_1, x_2, \dots, x_k)(x_{k+1}) \\ &= F(x_1, x_2, \dots, x_k)(x_{k+1}) \\ &= F(x_1, x_2, \dots, x_k, x_{k+1}), \end{aligned}$$

the last equation by the remark above, with  $F$  in place of  $D$  and  $k + 1$  in place of  $n$ .

In the remainder of this paragraph, we discuss partial fraction expansions of rational functions.

**36.4 Lemma:** *Let  $K$  be a field and let  $f(x)$  be a nonzero polynomial in  $K[x]$ . Let  $q(x), r(x)$  be two nonzero, relatively prime polynomials of positive degree in  $K[x]$ . Suppose  $\deg f(x) < \deg q(x)r(x)$  and suppose that  $f(x)$  is relatively prime to  $q(x)r(x)$ . Then there are uniquely determined nonzero polynomials  $a(x), b(x)$  in  $K[x]$  such that*

$$a(x)r(x) + b(x)q(x) = f(x), \quad \deg a(x) < \deg q(x), \quad \deg b(x) < \deg r(x).$$

**Proof:** We first prove the existence of  $a(x)$  and  $b(x)$ . Since  $q(x), r(x)$  are relatively prime, there are polynomials  $h(x), k(x)$  in  $K[x]$  with

$$h(x)r(x) + k(x)q(x) = 1.$$

Multiplying both sides of this equation by  $f(x)$  and putting  $A(x) = f(x)h(x), B(x) = f(x)k(x)$ , we obtain

$$A(x)r(x) + B(x)q(x) = f(x).$$

We now divide  $A(x)$  by  $q(x)$  and  $B(x)$  by  $r(x)$ :

$$\begin{aligned} A(x) &= s(x)q(x) + a(x), & a(x) &= 0 \text{ or } \deg a(x) < \deg q(x), \\ B(x) &= u(x)r(x) + b(x), & b(x) &= 0 \text{ or } \deg b(x) < \deg r(x). \end{aligned}$$

Thus

$$\begin{aligned} a(x)r(x) + b(x)q(x) &= (A(x) - s(x)q(x))r(x) + (B(x) - u(x)r(x))q(x) \\ &= (A(x)r(x) + B(x)q(x)) - (s(x) + u(x))q(x)r(x) \\ &= f(x) - (s(x) + u(x))q(x)r(x). \end{aligned}$$

We claim  $s(x) + u(x)$  is the zero polynomial in  $K[x]$ . Otherwise, we would have

$$\begin{aligned} \deg (s(x) + u(x)) &\geq 0, \\ \deg (s(x) + u(x))q(x)r(x) &\geq \deg q(x)r(x), \end{aligned}$$

and since by hypothesis  $\deg f(x) < \deg q(x)r(x)$ ,

$$\deg f(x) - (s(x) + u(x))q(x)r(x) \geq \deg q(x)r(x),$$

so that  $a(x)r(x) + b(x)q(x) \neq 0$ ; in particular, both  $a(x)$  and  $b(x)$  cannot be zero. Assume, without loss of generality, that  $a(x) \neq 0$  in case one of  $a(x), b(x)$  is zero and that  $\deg a(x)r(x) \geq \deg b(x)q(x)$  in case neither of them is zero. Then we get the contradiction

$$\deg [f(x) - (s(x) + u(x))q(x)r(x)] = \deg (a(x)r(x) + b(x)q(x))$$

$$\begin{aligned}
&\leq \deg a(x)r(x) \\
&= \deg a(x) + \deg r(x) \\
&< \deg q(x) + \deg r(x) \\
&= \deg q(x)r(x).
\end{aligned}$$

Thus  $s(x) + u(x)$ , and consequently  $(s(x) + u(x))q(x)r(x)$  is the zero polynomial in  $K[x]$ . This gives  $a(x)r(x) + b(x)q(x) = f(x)$ . It remains to show that  $a(x)$  and  $b(x)$  are distinct from the the zero polynomial in  $K[x]$ . Both of them cannot be 0, for then  $f(x)$  would be also 0, which it is not by hypothesis. If one of them is 0, say if  $a(x) = 0$ , then  $b(x) \neq 0$  and  $f(x) = b(x)q(x)$  would not be relatively prime to  $q(x)r(x)$  (because  $q(x)$  is of positive degree, so not a unit in  $K[x]$ ), against the hypothesis. This proves the existence of  $a(x)$ ,  $b(x)$ .

It remains to show the uniqueness of  $a(x)$  and  $b(x)$ . If we have also

$$a_1(x)r(x) + b_1(x)q(x) = f(x), \quad \deg a_1(x) < \deg q(x), \quad \deg b_1(x) < \deg r(x),$$

$$\begin{aligned}
\text{we obtain } 0 = f(x) - f(x) &= (a_1(x)r(x) + b_1(x)q(x)) - (a(x)r(x) + b(x)q(x)) \\
&= (a(x) - a_1(x))r(x) - (b_1(x) - b(x))q(x),
\end{aligned}$$

$$\text{so} \quad (a(x) - a_1(x))r(x) = (b_1(x) - b(x))q(x). \quad (*)$$

Hence

$$r(x) \mid (b_1(x) - b(x))q(x) \quad \text{in } K[x]$$

$$r(x) \mid b_1(x) - b(x) \quad \text{in } K[x] \text{ as } r(x) \text{ and } q(x) \text{ are relatively prime.}$$

Now  $b(x) \neq b_1(x)$  implies  $b(x) - b_1(x) \neq 0$  and this gives

$$\deg r(x) \leq \deg (b_1(x) - b(x)) \leq \max\{\deg b_1(x), \deg b(x)\} < \deg r(x),$$

a contradiction. Thus  $b(x) = b_1(x)$  and we get then  $a(x) = a_1(x)$  from (\*).

So  $a(x)$  and  $b(x)$  are uniquely determined.  $\square$

**36.5 Lemma:** *Let  $K$  be a field and let  $\frac{f(x)}{g(x)}$  be a nonzero rational function in  $K(x)$ , with  $\deg f(x) < \deg g(x)$ . Suppose that  $f(x)$  and  $g(x)$  are both monic and that  $f(x)$  is relatively prime to  $g(x)$ . Assume  $g(x) = q(x)r(x)$ , where  $q(x)r(x)$  are two relatively prime polynomials of positive degree in  $K[x]$ . Then there are uniquely determined nonzero polynomials  $a(x)$ ,  $b(x)$  in  $K[x]$  such that*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{q(x)r(x)} = \frac{a(x)}{q(x)} + \frac{b(x)}{r(x)}$$

and  $\deg a(x) < \deg q(x)$ ,  $\deg b(x) < \deg r(x)$ .

**Proof:** If  $\frac{f(x)}{g(x)}$  is a nonzero rational function in  $K(x)$ , then  $f(x)$  is a nonzero polynomial in  $K[x]$ , and  $f(x)$  is relatively prime to  $g(x) = q(x)r(x)$ . As  $f(x)$  and  $g(x)$  are monic, these conditions determine  $f(x)$  and  $g(x)$  uniquely. The polynomials  $q(x)$ ,  $r(x)$  are relatively prime and  $\deg f(x)$  is smaller than  $\deg q(x)r(x)$ . So the hypotheses of Lemma 36.4 are satisfied and therefore there are uniquely determined nonzero polynomials  $a(x), b(x)$  in  $K[x]$  such that

$$f(x) = a(x)r(x) + b(x)q(x),$$

and  $\deg a(x) < \deg q(x)$ ,  $\deg b(x) < \deg r(x)$ .

Dividing both sides of the equation above by  $g(x) = q(x)r(x)$ , we see that there are uniquely determined nonzero polynomials  $a(x), b(x)$  in  $K[x]$  such that

$$\frac{f(x)}{g(x)} = \frac{f(x)}{q(x)r(x)} = \frac{a(x)}{q(x)} + \frac{b(x)}{r(x)}$$

and  $\deg a(x) < \deg q(x)$ ,  $\deg b(x) < \deg r(x)$ . □

By induction on  $m$ , we obtain the following lemma.

**36.6 Lemma:** *Let  $K$  be a field and let  $\frac{f(x)}{g(x)}$  be a nonzero rational function in  $K(x)$ , with  $\deg f(x) < \deg g(x)$ . Suppose that  $f(x)$  and  $g(x)$  are both monic and that  $f(x)$  is relatively prime to  $g(x)$ . Assume  $g(x) = q_1(x)q_2(x) \cdots q_m(x)$ , where  $q_1(x), q_2(x), \dots, q_m(x)$  are pairwise relatively prime monic polynomials of positive degree in  $K[x]$ . Then there are uniquely determined nonzero polynomials  $a_1(x), a_2(x), \dots, a_m(x)$  in  $K[x]$  such that*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{q_1(x)q_2(x) \cdots q_m(x)} = \frac{a_1(x)}{q_1(x)} + \frac{a_2(x)}{q_2(x)} + \cdots + \frac{a_m(x)}{q_m(x)}$$

and  $\deg a_i(x) < \deg q_i(x)$  for all  $i = 1, 2, \dots, m$ .

**36.7 Lemma:** *Let  $K$  be a field and  $x$  an indeterminate over  $K$ . Let  $g(x)$  be a polynomial in  $K[x]$  of degree  $\geq 1$ . Then, for any  $f(x) \in K[x]$ , there are uniquely determined polynomials  $r_0(x), r_1(x), r_2(x), \dots, r_n(x)$  such that*

$$f(x) = r_0(x) + r_1(x)g(x) + r_2(x)g(x)^2 + \cdots + r_n(x)g(x)^n$$

and

$$r_i(x) = 0 \quad \text{or} \quad \deg r_i(x) < \deg g(x) \quad \text{for all } i = 1, 2, \dots, n.$$

**Proof:** From  $\deg g \geq 1$ , we know that  $g \neq 0$ . So we may divide  $f$  by  $g$  and obtain  $f = q_0g + r_0$ , where  $q_0, r_0 \in K[x]$ , with  $r_0 = 0$  or  $\deg r_0 < \deg g$ . Here  $q_0$  and  $r_0$  are uniquely determined by  $f$  and  $g$  (Theorem 34.4) and we have  $f = r_0 + q_0g$ . If  $q_0 = 0$ , we are done (with  $n = 0$ ). Otherwise, since  $f = q_0g + r_0$ ,  $\deg g \geq 1$  and  $r_0 = 0$  or  $\deg r_0 < \deg g$ , we have  $\deg q_0 < \deg f$  (Lemma 33.3). We now divide  $q_0$  by  $g$  and obtain  $q_0 = q_1g + r_1$ , where  $q_1, r_1 \in K[x]$ , with  $r_1 = 0$  or  $\deg r_1 < \deg g$ . Here  $q_1$  and  $r_1$  are uniquely determined by  $q_0$  and  $g$  (hence by  $f$  and  $g$ ) and  $f = r_0 + r_1g + q_1g^2$ . If  $q_1 = 0$ , we are done. Otherwise,  $\deg q_1 < \deg q_0$ . We then divide  $q_1$  by  $g$  and obtain  $q_1 = q_2g + r_2$ , where  $q_2, r_2 \in K[x]$ , with  $r_2 = 0$  or  $\deg r_2 < \deg g$ . Here  $q_2$  and  $r_2$  are uniquely determined by  $q_1$  and  $g$  (hence by  $f$  and  $g$ ) and  $f = r_0 + r_1g + r_2g^2 + q_2g^3$ . If  $q_2 = 0$ , we are done. Otherwise, we have  $\deg q_2 < \deg q_1$ . We continue this process. As the degrees of  $q_0, q_1, q_2, \dots$  get smaller and smaller, this process cannot go on indefinitely. Sooner or later, we will meet a  $q_n$  equal to  $0 \in K[x]$ . Then, with uniquely determined  $r_0, r_1, r_2, \dots, r_n$ , we have  $f = r_0 + r_1g + r_2g^2 + \cdots + r_n g^n$ , where  $r_i(x) = 0$  or  $\deg r_i < \deg g$  for all  $i = 1, 2, \dots, n$ .

□

In the situation of Lemma 36.7, the unique expression

$$f = r_0 + r_1g + r_2g^2 + \cdots + r_n g^n$$

of  $f(x)$ , where  $r_i(x) = 0$  or  $\deg r_i < \deg g$  for all  $i = 1, 2, \dots, n$ , is called the *g-adic expansion of f*.

**36.8 Theorem:** Let  $K$  be a field and  $\frac{p(x)}{q(x)}$  a nonzero rational function in  $K(x)$ , where  $p(x), q(x) \in K[x]$  are relatively prime in  $K[x]$ . Let  $u$  be the leading coefficient of  $q(x)$  and let  $q(x) = ug_1(x)^{m_1}g_2(x)^{m_2} \cdots g_t(x)^{m_t}$  be the decomposition of  $q(x)$  into polynomials irreducible over  $K$ , where  $g_i(x)$  are monic. Then there are uniquely determined polynomials  $G(x)$ ,

$a_1^{(1)}(x), a_2^{(1)}(x), \dots, a_{m_1}^{(1)}(x), a_1^{(2)}(x), a_2^{(2)}(x), \dots, a_{m_2}^{(2)}(x), \dots, a_1^{(t)}(x), a_2^{(t)}(x), \dots, a_{m_t}^{(t)}(x)$  in  $K[x]$  such that

$$\begin{aligned} \frac{p(x)}{q(x)} = G(x) &+ \frac{a_1^{(1)}(x)}{g_1^{m_1}(x)} + \frac{a_2^{(1)}(x)}{g_2^{m_2}(x)} + \cdots + \frac{a_{m_1}^{(1)}(x)}{g_1^{m_1}(x)} \\ &+ \frac{a_1^{(2)}(x)}{g_2^{m_2}(x)} + \frac{a_2^{(2)}(x)}{g_2^{m_2}(x)} + \cdots + \frac{a_{m_2}^{(2)}(x)}{g_2^{m_2}(x)} \\ &+ \cdots \cdots \\ &+ \frac{a_1^{(t)}(x)}{g_t^{m_t}(x)} + \frac{a_2^{(t)}(x)}{g_t^{m_t}(x)} + \cdots + \frac{a_{m_t}^{(t)}(x)}{g_t^{m_t}(x)} \end{aligned}$$

and  $\deg a_i^{(k)}(x) \leq \deg g_k(x)$  or  $a_i^{(k)}(x) = 0$  for all  $i$  and  $k$ .

**Proof:** We divide  $p(x)$  by  $q(x)$  and find unique polynomials  $G(x), H(x)$  in  $K[x]$  with  $p(x) = q(x)G(x) + H(x)$ ,  $\deg H(x) < \deg q(x)$  or  $H(x) = 0$ . In the latter case, everything is proved ( $a_i^{(k)}(x) = 0$  for all  $i$  and  $k$ ). If  $H(x) \neq 0$ , let  $v$  be the leading coefficient of  $H(x)$  and put  $c = v/u$ . Then  $H(x)$  and  $q(x)$  are relatively prime (since  $p(x)$  and  $q(x)$  are). We have  $H(x) = vh(x)$ , where  $h(x)$  is monic, relatively prime to  $q(x)$  and

$$\frac{p(x)}{q(x)} = G(x) + c \frac{h(x)}{q(x)}$$

with  $\deg h(x) < \deg q(x)$ . We may use Lemma 36.6 and get uniquely determined nonzero polynomials  $b_1(x), b_2(x), \dots, b_t(x)$  in  $K[x]$  such that

$$\frac{h(x)}{q(x)} = \frac{b_1(x)}{g_1(x)^{m_1}} + \frac{b_2(x)}{g_2(x)^{m_2}} + \cdots + \frac{b_t(x)}{g_t(x)^{m_t}}$$

and  $\deg b_k(x) < \deg g_k(x)^{m_k}$  for all  $k = 1, 2, \dots, t$ . We put  $f_k(x) = cb_k(x)$ . Then

$$\frac{p(x)}{q(x)} = G(x) + \frac{f_1(x)}{g_1(x)^{m_1}} + \frac{f_2(x)}{g_2(x)^{m_2}} + \cdots + \frac{f_t(x)}{g_t(x)^{m_t}}$$

and, since  $c$  is uniquely determined by  $p(x)$  and  $q(x)$ , the polynomials  $f_k(x)$  are also uniquely determined. Since

$$\deg f_k(x) = \deg b_k(x) < \deg g_k(x)^{m_k},$$

in the  $g_k(x)$ -adic expansion

$$f_k(x) = r_0(x) + r_1(x)g_k(x) + r_2(x)g_k(x)^2 + \cdots + r_n(x)g_k(x)^n$$

of  $f_k(x)$ , the polynomials  $r_s(x) = 0$  for  $s \geq m_k$ . So let

$$f_k(x) = a_1^{(k)}(x)g_k(x)^{m_k-1} + a_2^{(k)}(x)g_k(x)^{m_k-2} + \cdots + a_{m_k-1}^{(k)}(x)g_k(x) + a_{m_k}^{(k)}(x)$$

be the  $g_k(x)$ -adic expansion of  $f_k(x)$ . The polynomials  $a_1^{(k)}, a_2^{(k)}, \dots, a_{m_k}^{(k)}$  in  $K[x]$  are uniquely determined and  $\deg a_i^{(k)} < \deg g_k(x)$  or  $a_i^{(k)} = 0$  for all  $i = 1, 2, \dots, m_k$ . Hence, for all  $k = 1, 2, \dots, t$ , there holds

$$\frac{f_k(x)}{g_k(x)^{m_k}} = \frac{a_1^{(k)}(x)}{g_k^1(x)} + \frac{a_2^{(k)}(x)}{g_k^2(x)} + \cdots + \frac{a_{m_k}^{(k)}(x)}{g_k^{m_k}(x)}$$

and this completes the proof.

The equation

$$\begin{aligned} \frac{p(x)}{q(x)} = G(x) &+ \frac{a_1^{(1)}(x)}{g_1^1(x)} + \frac{a_2^{(1)}(x)}{g_1^2(x)} + \cdots + \frac{a_{m_1}^{(1)}(x)}{g_1^{m_1}(x)} \\ &+ \frac{a_1^{(2)}(x)}{g_2^1(x)} + \frac{a_2^{(2)}(x)}{g_2^2(x)} + \cdots + \frac{a_{m_2}^{(2)}(x)}{g_2^{m_2}(x)} \\ &+ \cdots \\ &+ \frac{a_1^{(t)}(x)}{g_t^1(x)} + \frac{a_2^{(t)}(x)}{g_t^2(x)} + \cdots + \frac{a_{m_t}^{(t)}(x)}{g_t^{m_t}(x)} \end{aligned}$$

in Theorem 36.8 is known as the *expansion of  $\frac{p(x)}{q(x)}$  in partial fractions*.

## Exercises

1. Let  $K$  be a field. For any nonzero rational function  $\frac{f}{g}$  in  $K(x)$ , we define the *degree of  $\frac{f}{g}$* , denoted by  $\deg \frac{f}{g}$ , by  $\deg \frac{f}{g} = \deg f - \deg g$ . Prove that the degree of a rational function is well defined. Can you extend the degree assertions in Lemma 33.3 to rational functions?
2. Let  $K$  be a field. For any rational function  $\frac{f}{g}$  in  $K(x)$ , we define the *derivative of  $\frac{f}{g}$* , denoted by  $\left(\frac{f}{g}\right)'$ , by declaring

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Prove that differentiation is well defined, i.e., prove that  $\frac{f}{g} = \frac{a}{b}$  implies

$$\left(\frac{f}{g}\right)' = \left(\frac{a}{b}\right)'.$$

3. Extend Lemma 35.15 and Lemma 35.16 to derivatives of rational functions in one indeterminate over a field.

4. Expand  $\frac{2x^3 + 3x^2 + 8x + 6}{(x^3 + 3x + 3)(x^2 + 2x + 3)} \in \mathbb{Q}(x)$  and

$$\frac{4x^3 + 3x^2 + x + 2}{x^5 + 4x^4 + 4x^3 + 2x + 2} \in \mathbb{Z}_5(x)$$

in partial fractions.

5. Let  $K$  be a field and let  $a_1, a_2, \dots, a_m$  be pairwise distinct elements in  $K$ . Put  $g(x) = (x - a_1)(x - a_2) \dots (x - a_m)$  and let  $f(x)$  be a nonzero polynomial in  $K[x]$  with  $\deg f(x) < m$ . Show that

$$\frac{f(x)}{g(x)} = \sum_{i=1}^m \frac{f(a_i)/g'(a_i)}{x - a_i}.$$