CHAPTER 4

Vector Spaces

§39
Definition and Examples

The term "vector" is familiar to the reader from Physics. Such physical magnitudes as displacement, force, torque, momentum etc. are vectors. Vectors can be added (by the parallelogram law) and multiplied by real numbers, which are called scalars in this context. In this chapter, we introduce systems of objects which can be added and multiplied by scalars.

39.1 Definition: Let K be a field and let $(V,+)$ be an (additively written) abelian group. Suppose that, to each pair (α,v) in $K \times V$, there corresponds a unique element of V, denoted by $\alpha.v$, such that the following equations hold for all $\alpha,\beta \in K$, $v,w \in V$:

(1) $\alpha.(v + w) = \alpha.v + \alpha.w$

(2) $(\alpha + \beta).v = \alpha.v + \beta.w$

(3) $\alpha.(\beta.v) = (\alpha\beta).v$

(4) $1.v = v$ (1 is the identity of K).

In this case, the ordered quadruple $(V,+,K,\cdot)$ is called a *vector space over* K, or a K-*vector space*. The elements of K are called *scalars*, and K is called the *field of scalars* of the vector space $(V,+,K,\cdot)$. The elements of V

are called *vectors*. The mapping $(\alpha, v) \to \alpha . v$ from $K \times V$ into V will be called *multiplication by scalars.*

From now on, the term "vector" will mean an element of a vector space. We will see vectors which do not resemble the vectors of physics in any way.

At the cost of some stylistic clumsiness in the formulation of many statements, we shall refer to the mapping $(\alpha, v) \to \alpha . v$ as multiplication by scalars, not as scalar multiplication. This is what the mapping really is. It is multiplication of vectors by scalars, not a multiplication whose results are scalars. We will usually omit $\cdot$ and write αv instead of $\alpha . v$.

Strictly speaking, a vector space is an ordered quadruple $(V, +, K, \cdot)$. However, as in the case of groups and rings, we shall usually refer to the *set* V as a vector space over K. If the field of scalars is fixed throughout a discussion, we shall speak of vector spaces, without reference to the field of scalars.

It will be convenient to think of a vector space as an abelian group with an additional structure on it supplied by the multiplication by scalars. The wording of Definition 39.1 was chosen to emphasize this point of view.

39.2 Examples: (a) Let $V = \mathbb{R} \oplus \mathbb{R}$ be the direct sum of two copies of $\mathbb{R}$ and let $K = \mathbb{R}$. We define multiplication by scalars in the most natural way: $\alpha(\beta,\gamma) = (\alpha\beta,\alpha\gamma)$ (for all $\alpha \in \mathbb{R}$, $(\beta,\gamma) \in V$).
It is easily seen that V is an $\mathbb{R}$-vector space.

(b) The same construction can be carried out with n-tuples of elements from any field K. Let K be a field and let $V = K \oplus K \oplus \ldots \oplus K$ be the direct sum of n copies of K, which is an abelian group under componentwise addition. We define multiplication by scalars also componentwise:

$$\alpha(\beta_1,\beta_2, \ldots,\beta_n) = (\alpha\beta_1,\alpha\beta_2, \ldots,\alpha\beta_n) \quad \text{(for all } \alpha \in K, (\beta_1,\beta_2, \ldots,\beta_n) \in V).$$

It is easily verified that V is a K-vector space. It will be designated by K^n, and will be called the *K-vector space of n-tuples.*

(c) Let V be the set of all real-valued functions defined on the interval [0,1]. For any two functions f,g in V, we define a new function $f+g$ in V by $$(f+g)(x) = f(x) + g(x) \qquad \text{for all } x \in [0,1].$$
V is an abelian group under this addition (called the *pointwise addition* of functions). Now let $K = \mathbb{R}$ and define a *pointwise multiplication* by scalars: $$(\alpha f)(x) = \alpha f(x) \qquad \text{for all } x \in [0,1],\ \alpha \in \mathbb{R}.$$
Then V is a vector space over $\mathbb{R}$ (cf. Example 29.2(i)).

When we put $$(\alpha f)(x) = \alpha f(x) \qquad \text{for all } x \in [0,1]\ \alpha \in \mathbb{C},$$
then V would not be a vector space over $\mathbb{C}$, because αf would not belong to V for all $\alpha \in \mathbb{C}$, $f \in V$, as the function αf is not real-valued when α is a complex number with a nonzero imaginary part.

(d) Let K be a field and let $K[x]$ be the ring of all polynomials over K. Let us forget that we can multiply two polynomials and concentrate on the fact that we can add them and multiply them by the elements of K (which are polynomials of degree zero, or the zero polynomial). It is easily seen that $K[x]$ is a K-vector space.

(e) Let n be a fixed natural number. Let K be a field and let V be the set of all polynomials over K which have degree n. Is V a vector space over K? No, because the sum of two polynomials of degree n is not always a polynomial of degree n (when the leading coefficients are opposites of each other). On the other hand, the set consisting of the zero polynomial and of all polynomials over K whose degrees are less than or equal to n is a vector space over K.

(f) Let V be a vector space over a field K, and let K_1 be a field contained in K (in this case, K_1 is called a *subfield of* K). Then V is a vector space over K_1, too, since the requirements in Definition 39.1 are satisfied by the elements of K_1 if they are satisfied by the elements of K.

(g) Let K be a field. When we define the multiplication by scalars as the multiplication in the field K, then K becomes a vector space over K. The conditions in Definition 39.1 are simply the distributivity laws, the associativity of multiplication and the very definition of the identity element in K.

(h) It follows from Example 39.2(f) and Example 39.2(g) that, if K_1 and K are fields such that $K_1 \subseteq K$, then K is a vector space over K_1: any field is a vector space over its subfields. For instance, $\mathbb{C}$ is a vector space over $\mathbb{R}$, and $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

39.3 Remarks: (1) A vector space is an abelian group and has an identity element, which we call zero and denote by 0. The underlying field K has a zero element, too, which is also denoted by 0. The reader should carefully distinguish between these zeroes. One of them is a vector, the other is a scalar. The vector zero is sometimes denoted by $\vec{0}$.

(2) Multiplication by scalars is a mapping from $K \times V$ into V, hence it is *not* a binary operation on V unless $K = V$. This feature distinguishes vector spaces from groups and rings. Multiplication and addition are binary operations on groups and rings.

Some basic facts are collected in the next lemma.

39.4 Lemma: *Let V be a vector space over a field K. For all $\alpha,\beta \in K$ and for all $u,v,w \in V$, the following hold.*
(1) $0 + v = v$.
(2) $-v + v = 0$.
(3) $-0 = 0$ (vector zero).
(4) $u + v = u + w$ *implies* $v = w$.
(5) $\alpha 0 = 0$.
(6) $0v = 0$.
(7) $\alpha(-v) = -(\alpha v) = (-\alpha)v$; *in particular*, $-1.v = -v$.
(8) $(\alpha - \beta)v = \alpha v - \beta v$.
(9) $\alpha(v - w) = \alpha v - \alpha w$.
(10) $\alpha v = 0$ *implies* $\alpha = 0$ or $v = 0$.
(11) $\alpha v = \beta v$ *implies* $\alpha = \beta$ or $v = 0$.
(12) $\alpha v = \alpha w$ *implies* $\alpha = 0$ or $v = w$.

Proof: (1),(2),(3),(4) hold in any group (Lemma 7.3, Lemma 8.2), also in the abelian group $(V,+)$.

(5) We are to prove $\alpha 0 = 0$ (vector zero). We observe

$$\alpha 0 + 0 = \alpha 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0,$$

hence $\alpha 0 = 0$ by (4).

(6) We are to prove $0v = 0$ (on the left hand side, we have the scalar zero, on the right hand side, the vector zero). We observe

$$0v + 0 = 0v = (0 + 0)v = 0v + 0v,$$

hence $0v = 0$ by (4).

(7) We have $\qquad 0 = \alpha 0 = \alpha\big(v + (-v)\big) = \alpha v + \alpha(-v)$

and $\qquad 0 = 0v = \big(\alpha + (-\alpha)\big)v = \alpha v + (-\alpha)v$

by (5) and (6), so $(-\alpha)v$ and $\alpha(-v)$ are the opposite of αv. Thus

$$\alpha(-v) = -(\alpha v) = (-\alpha)v.$$

(8) We are to show $(\alpha - \beta)v = \alpha v - \beta v$. Here $\alpha - \beta$ is an abbreviation for $\alpha + (-\beta)$ in K, and $\alpha v - \beta v$ is an abbreviation for $\alpha v + \big(-(\beta v)\big)$ in V. We have indeed: $\quad (\alpha - \beta)v = \big(\alpha + (-\beta)\big)v = \alpha v + (-\beta)v = \alpha v + \big(-(\beta v)\big) = \alpha v - \beta v.$

(9) $\alpha(v - w) = \alpha\big(v + (-w)\big) = \alpha v + \alpha(-w) = \alpha v + \big(-(\alpha w)\big) = \alpha v - \alpha w.$

(10) Assume $\alpha v = 0$. If $\alpha \neq 0$, then α^{-1} exists in K and we get

$$v = 1v = (\alpha^{-1}\alpha)v = \alpha^{-1}(\alpha v) = \alpha^{-1}0 = 0.$$

(11) This follows from (8) and (10).

(12) This follows from (9) and (10). □

39.5 Lemma: *Let V be a vector space over a field K. Then, for all $\alpha_1,\alpha_2,\ldots,\alpha_n$, α in K and $v_1,v_2,\ldots,v_n,v$ in V, there hold*

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_n)v = \alpha_1 v + \alpha_2 v + \cdots + \alpha_n v$$

and

$$\alpha(v_1 + v_2 + \cdots + v_n) = \alpha v_1 + \alpha v_2 + \cdots + \alpha v_n.$$

Proof: This follows by induction on n. The details are left to the reader. □

Just as there may be different group structures on a set, there may also be different vector space structures on a set. Here is an example.

39.6 Example: Let $V := \mathbb{C} \oplus \mathbb{C}$. We define a multiplication $\circ$ of the elements of V by complex numbers by declaring

$$c \circ (a,b) = (\overline{c}a,\overline{c}b) \qquad \text{for all } c \in \mathbb{C},\ (a,b) \in V.$$

This multiplication makes the abelian group V into a $\mathbb{C}$-vector space:

$$
\begin{array}{lll}
(1) & c \circ [(a,b) + (d,e)] & = c \circ (a + d, b + e) \\
& & = \big(\overline{c}(a + d), \overline{c}(b + e)\big) \\
& & = (\overline{c}a + \overline{c}d, \overline{c}b + \overline{c}e) \\
& & = (\overline{c}a, \overline{c}b) + (\overline{c}d, \overline{c}e) \\
& & = c \circ (a,b) \ + \ c \circ (d,e), \\
(2) & (c + f) \circ (a,b) & = \big(\overline{(c + f)}a, \overline{(c + f)}b\big) \\
& & = (\overline{c}a + \overline{f}a, \ \overline{c}b + \overline{f}b) \\
& & = (\overline{c}a, \overline{c}b) + (\overline{f}a, \overline{f}b) \\
& & = c \circ (a,b) \ + \ f \circ (a,b), \\
(3) & (cf)(ab) & = (\overline{cf}a, \overline{cf}b) \\
& & = (\overline{c}\,\overline{f}a, \overline{c}\,\overline{f}b) \\
& & = c \circ (\overline{f}a, \overline{f}b) \\
& & = c \circ \big(f \circ (a,b)\big), \\
(4) & 1 \circ (a,b) & = (\overline{1}a, \overline{1}b) = (1a, 1b) = (a,b)
\end{array}
$$

for all $(a,b),(d,e) \in V$, $c,f \in \mathbb{C}$. Thus $(V,+,\mathbb{C},\circ)$ is a vector space, with the same set V, the same addition + on V, the same underlying field $\mathbb{C}$ as the vector space $(V,+,\mathbb{C},\cdot)$ of Example 39.2(b) (in case $K = \mathbb{C}$, $n = 2$), but $(V,+,\mathbb{C},\circ)$ is distinct from $(V,+,\mathbb{C},\cdot)$ since the multiplication by scalars in these vector spaces are different.

Exercises

1. Determine whether $\mathbb{R} \times \mathbb{R}$ is an $\mathbb{R}$-vector space when
$$(a,b) + (c,d) = (a + c, b + d), \quad a(c,d) = (c,d) \quad \text{for all } a,b,c,d \in \mathbb{R}.$$

2. Determine whether $\mathbb{Q} \times \mathbb{Q}$ is a $\mathbb{Q}$-vector space when
$$(a,b) + (c,d) = (a + c, 0), \quad a(c,d) = (ac, ad) \quad \text{for all } a,b,c,d \in \mathbb{Q}.$$

3. Determine whether $\mathbb{Z}_7 \times \mathbb{Z}_7$ is a $\mathbb{Z}_7$-vector space when
$$(a,b) + (c,d) = (a + c, b + d), \quad a(c,d) = (ac, 0) \quad \text{for all } a,b,c,d \in \mathbb{Z}_7.$$

4 Determine whether the set S of all sequences of real numbers is a vector space over $\mathbb{R}$ if addition and multiplication by scalars are defined by $(a_n) + (b_n) = (a_n + b_n), \quad a(b_n) = (ab_n)$
for all $(a_n),(b_n) \in S$, $a \in \mathbb{R}$ (here (a_n) is the sequence $a_1, a_2, a_3, \ldots$).

5. If $q \in \mathbb{N}$ and K is a field of q elements, how many elements does K^n have?

6. Construct a vector space with exactly four elements.