§40
Subspaces

Just as we defined subgroups and subrings, we will define sub(vector space)s. We contract this awkward expression into "subspace".

40.1 Definition: Let V be a vector space over a field K. A nonempty subset W of V is called a *subspace of* V if W itself is a K-vector space (under the addition and multiplication by scalars inherited from V).

A subspace W of V is an abelian group, thus a subgroup of $(V,+)$. Also, products by scalars of the element of W belong to W, so that $\alpha w \in W$ whenever $\alpha \in K$ and $w \in W$. Conversely, any subgroup W of V such that $\alpha w \in W$ for all $\alpha \in K$ and $w \in W$ is easily seen to be a subspace of V, for the conditions in Definition 39.1 are automatically satisfied for all elements of W if they are satisfied for all elements of V. Thus W is a subspace of V if and only if

(1) W is a subgroup of V under addition,

(ii) if $\alpha \in K$ and $w \in W$, then $\alpha w \in W$ (i.e.,W is closed under multiplication by scalars).

Here (1) embraces two conditions: (i) W is closed under addition, (ii′) for any $w \in W$, the opposite $-w$ of w also belongs to W. Thus W is a subspace if and only if (i),(ii′) and (ii) hold. One checks easily that (ii) implies (ii′): if (ii) holds and $w \in W$, then $(-1)w \in W$, hence $-w \in W$ by Lemma 39.4(7), so (ii′) holds. Thus (ii′) is superfluous. We proved the following lemma.

40.2 Lemma (Subspace criterion): *Let V be a vector space over a field K and let W be a nonempty subset of V. Then W is a subspace of V if and only if*

(i) $w_1 + w_2 \in W$ *for all* $w_1,w_2 \in W$,

(ii) $\alpha w \in W$ *for all* $\alpha \in K$, $w \in W$. □

So a nonempty subset of a vector space V is a subspace of V if and only if it is closed under addition and multiplication by scalars. The two closure properties of Lemma 40.2 can be combined to a single one. When (i) and (ii) of Lemma 40.2 hold, then

$$\alpha w_1 + \beta w_2 \in W \qquad \text{for all } \alpha,\beta \in K,\ w_1,w_2 \in W \qquad (*)$$

since αw_1, $\beta w_2 \in W$ by (ii) and $\alpha w_1 + \beta w_2 \in W$ by (i). Conversely, if (*) holds, then, choosing $\alpha = 1$, $\beta = 1$, we see that (i) holds and, choosing $\beta = 0$, we see that (ii) holds. Thus (i) and (ii) are together equivalent to (*). Then we obtain another version of Lemma 40.2.

40.3 Lemma (Subspace criterion): *Let V be a vector space over a field K and let W be a nonempty subset of V. Then W is a subspace of V if and only if*

$$\alpha w_1 + \beta w_2 \in W \qquad \textit{for all } \alpha,\beta \in K,\ w_1,w_2 \in W. \qquad \square$$

The expression $\alpha w_1 + \beta w_2 \in W$ is said to be a linear combination of the vectors w_1,w_2. More generally, we have the

40.4 Definition: Let $v_1,v_2, \ldots ,v_n$ be finitely many (not necessarily distinct) vectors of a vector space V over a field K. A vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n,$$

where $\alpha_1,\alpha_2, \ldots ,\alpha_n \in K$, is called a *$K$-linear combination* of the vectors $v_1,v_2, \ldots ,v_n$. (If the underlying field K is clear from the context, we use the term "linear combination", without mentioning K.)

40.5 Lemma: *Let V be a vector space over a field K and let W be a subspace of V. If $w_1,w_2, \ldots ,w_m$ are vectors in W, then every K-linear combination of these vectors belongs to W.*

Proof: This follows from Lemma 40.3 by induction on m. $\square$

40.6 Examples: (a) Let V be any vector space. Then $\{0\}$ and V are subspaces of V.

(b) Consider the vector space K^3 over a field K and put

$$W = \{(\lambda,\mu,\nu) \in K^3 \colon \nu = 0\} \subseteq K^3.$$

If $(\lambda_1,\mu_1,0)$ and $(\lambda_2,\mu_2,0)$ are arbitrary vectors in W and if α,β are arbitrary scalars, then $\alpha(\lambda_1,\mu_1,0) + \beta(\lambda_2,\mu_2,0) = (\alpha\lambda_1 + \beta\lambda_2,\alpha\mu_1 + \beta\mu_2,0)$ belongs to W. By Lemma 40.3, W is a subspace of K^3.

(c) Consider the vector space K^3 over a field K and put

$$U = \{(\lambda,\mu,\nu) \in K^3 \colon \nu = 1\} \subseteq K^3.$$

Then $(0,0,1) \in U$, $(1,0,1) \in U$, but $(0,0,1) + (1,0,1) = (1,0,1+1) \notin U$ (why?) and U is not closed under addition. So U is not a subspace of K^3.

(d) Consider $\mathbb{R}^3$ over $\mathbb{R}$ and let

$$A = \{(\lambda,\mu,\nu) \in \mathbb{R}^3 \colon \nu \geqslant 0\} \subseteq \mathbb{R}^3.$$

If (λ_1,μ_1,ν_1) and (λ_2,μ_2,ν_2) are in A, then $\nu_1 \geqslant 0$, $\nu_2 \geqslant 0$, so $\nu_1 + \nu_2 \geqslant 0$ and

$$(\lambda_1,\mu_1,\nu_1) + (\lambda_2,\mu_2,\nu_2) = (\lambda_1 + \lambda_2,\mu_1 + \mu_2,\nu_1 + \nu_2)$$

belongs to A. Thus A is closed under addition. However, A is not a subspace of $\mathbb{R}^3$, since, for instance, $(0,0,1) \in A$ but $(-1)(0,0,1) \notin U$. This example shows that a subset of a vector space can be closed under addition without being closed under multiplication by scalars.

(e) Consider the vector space K^2 over a field K and let

$$M = \{(\lambda,\mu) \in K^2 \colon \lambda = 0 \text{ or } \mu = 0\} \subseteq K^2.$$

If $\alpha \in K$ and $(\lambda,\mu) \in M$, then $\lambda = 0$ or $\mu = 0$, so $\alpha\lambda = 0$ or $\alpha\mu = 0$, so $\alpha(\lambda,\mu) = (\alpha\lambda,\alpha\mu)$ belongs to M. Thus A is closed under multiplication by scalars. However, M is not a subspace of K^2, since, for instance, $(1,0), (0,1) \in M$, but $(1,0) + (0,1) \notin M$. This example shows that a subset of a vector space can be closed under multiplication by scalars without being closed under addition.

(f) Consider the vector space K^2 over a field K and let γ,δ be two arbitrary but fixed elements of K. Put

$R = \{(\lambda,\mu) \in K^2 \colon \gamma\lambda + \delta\mu = 0\} \subseteq K^2$. Then R is a subspace of K^2:

(i) If $(\lambda_1,\mu_1), (\lambda_2,\mu_2) \in R$, then $\gamma\lambda_1 + \delta\mu_1 = 0 = \gamma\lambda_2 + \delta\mu_2$, so $(\gamma\lambda_1 + \delta\mu_1) + (\gamma\lambda_2 + \delta\mu_2) = 0$, so $\gamma(\lambda_1 + \lambda_2) + \delta(\mu_1 + \mu_2) = 0$, so $(\lambda_1,\mu_1) + (\lambda_2,\mu_2) = (\lambda_1 + \lambda_2, \mu_1 + \mu_2) \in R$.

(ii) If $\alpha \in K$ and $(\lambda,\mu) \in R$, then $\gamma\lambda + \delta\mu = 0$, so $\gamma\alpha\lambda + \delta\alpha\mu = 0$, so $\alpha(\lambda,\mu) = (\alpha\lambda,\alpha\mu) \in R$.

(g) Let V be a vector space over a field K and let $\{W_i : i \in I\}$ be a collection of subspaces of V. Then their intersection $W := \bigcap_{i \in I} W_i$ is a subspace of V. First of all, this intersection is not empty, since $0 \in W_i$ for all $i \in I$. Also, if $\alpha,\beta \in K$ and $w_1, w_2 \in W$, then $\alpha,\beta \in K$ and $w_1, w_2 \in W_i$ for all $i \in I$, so $\alpha w_1 + \beta w_2 \in W_i$ for all $i \in I$, so $\alpha w_1 + \beta w_2 \in W$.

(h) Let V be the $\mathbb{R}$-vector space of all real-valued functions defined on [0,1] (See Example 39.2(c)) and let α be a fixed number in [0,1]. We put

$$T_\alpha = \{f \in V : f \text{ is continuous at } \alpha\}.$$

It is known from analysis that, if f and g are functions, continuous at α, then $f + g$ is also continuous at α. If f is continuous at α and $\beta \in \mathbb{R}$, then βf is continuous at α. Hence T_α is a subspace of V.

(i) Let V be the $\mathbb{R}$-vector space of all real-valued functions defined on [0,1]. We put

$$C([0,1]) = \{f \in V : f \text{ is continuous on } [0,1]\}.$$

We know from analysis that, if f and g are functions, continuous on [0,1], then $f + g$ is also continuous on [0,1]. If f is continuous at α, and $\beta \in \mathbb{R}$, then βf is continuous on [0,1]. Hence $C([0,1])$ is a subspace of V. This conclusion can be drawn also by observing that $C([0,1]) = \bigcap_{\alpha \in [0,1]} T_\alpha$ and appealing to Example 40.6(g) and Example 40.6(h).

(j) Let $C^1([0,1]) = \{f \in C([0,1]) : f' \text{ exists and is continuous on } [0,1]\}$. $C^1([0,1])$ is a nonempty subset of $C([0,1])$. If $\alpha,\beta \in \mathbb{R}$ and $f,g \in C^1([0,1])$, then, as is well known from analysis, $(\alpha f + \beta g)'$ exists, is equal to $\alpha f' + \beta g'$ and is continuous on [0,1]. Hence $\alpha f + \beta g \in C^1([0,1])$ and therefore $C^1([0,1])$ is a subspace of $C([0,1])$.

Similarly, for $k \in \mathbb{N}$, we put

$$C^k([0,1]) = \{f \in C([0,1]) : f^{(k)} \text{ exists and is continuous on } [0,1]\}.$$

Since the existence of the k-th derivative of f implies the existence and continuity of the first $k - 1$ derivatives $f', f'', \ldots, f^{(k-1)}$, we see $C^k([0,1])$ is a subset of $C^{k-1}([0,1])$. From the formula

$$(\alpha f + \beta g)^{(k)} = \alpha f^{(k)} + \beta g^{(k)} \qquad (\alpha,\beta \in \mathbb{R},\ f,g \in C^k([0,1]))$$

it is easily seen that $C^k([0,1])$ is a subspace of $C([0,1])$ and of $C^{k-1}([0,1])$.

We write $C^\infty([0,1]) = \bigcap_{k \in \mathbb{N}} C^k([0,1])$. From Example 40.6(g), we infer that $C^\infty([0,1])$ is also a subspace of $C([0,1])$ and of each $C^k([0,1])$.

(k) Let $p(x)$ and $q(x)$ be continuous functions, defined on $[0,1]$. We write

$$L = \{f \in \boldsymbol{C}^2([0,1]) \colon f''(x) + p(x)f'(x) + q(x)f(x) = 0 \text{ for all } x \in [0,1]\}.$$

L is a nonempty subset of $\boldsymbol{C}^2([0,1])$. If $\alpha,\beta \in \mathbb{R}$ and $f,g \in L$, then

$$\begin{aligned}
&(\alpha f + \beta g)''(x) + p(x)(\alpha f + \beta g)'(x) + q(x)(\alpha f + \beta g)(x)\\
&= \alpha f''(x) + \beta g''(x) + p(x)\alpha f'(x) + p(x)\beta g'(x) + q(x)\alpha f(x) + q(x)\beta g(x)\\
&= \alpha\big(f''(x) + p(x)f'(x) + q(x)f(x)\big) + \beta\big(g''(x) + p(x)g'(x) + q(x)g(x)\big)\\
&= \alpha 0 + \beta 0 = 0
\end{aligned}$$

for all $x \in [0,1]$, so $\alpha f + \beta g \in L$. Thus L is a subspace of $\boldsymbol{C}^2([0,1])$.

(l) So far, we spoke of subspaces without referring to the underlying field. Sometimes it might be necessary to mention the underlying field. Let $V = \mathbb{C}^2$ be the $\mathbb{C}$-vector space of ordered pairs of complex numbers. Then V is an $\mathbb{R}$-vector space, too (Example 39.2(f)). We put

$$W = \{(\lambda,\overline{\lambda}) \colon \lambda \in \mathbb{C}\} = \{(\lambda,\mu) \colon \lambda \in \mathbb{C},\ \mu = \overline{\lambda}\},$$

where $\overline{}$ denotes complex conjugation. If (λ,μ), $(\nu,\rho) \in W$ and $\alpha,\beta \in \mathbb{R}$, then $\mu = \overline{\lambda}$ and $\rho = \overline{\nu}$, so $\alpha(\lambda,\mu) + \beta(\nu,\rho) = (\alpha\lambda + \beta\nu,\ \alpha\mu + \beta\rho)$ with

$$\begin{aligned}
\overline{\alpha\lambda + \beta\nu} &= \overline{\alpha\lambda} + \overline{\beta\nu}\\
&= \overline{\alpha}\,\overline{\lambda} + \overline{\beta}\,\overline{\nu}\\
&= \alpha\overline{\lambda} + \beta\overline{\nu} \qquad\qquad (c)\\
&= \alpha\mu + \beta\rho,
\end{aligned}$$

and $\alpha(\lambda,\mu) + \beta(\nu,\rho) \in W$. Thus W is a subspace of the $\mathbb{R}$-vector space V. However, W is not a subspace of the $\mathbb{C}$-vector space V, for the critical equation (c) need not be true when α,β are complex numbers (with nonzero imaginary parts). We may say W is an $\mathbb{R}$-subspace of V, but not a $\mathbb{C}$-subspace of V.

40.7 Theorem: *Let V be a vector space over a field K and let $A = \{v_1,v_2,\dots,v_n\}$ be a finite nonempty subset of V. Then the set*

$$W = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n\}$$

of all linear combinations of the vectors $v_1,v_2,\dots,v_n$ is a subspace of V.

Proof: Since A is not empty, $W \neq \emptyset$. If $\alpha,\beta \in K$ and $u,w \in W$, then

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \qquad w = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

with suitable $\alpha_1,\alpha_2,\dots,\alpha_n,\ \beta_1,\beta_2,\dots,\beta_n \in K$ and

$$\begin{aligned}
\alpha u + \beta w &= \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) + \beta(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n)\\
&= (\alpha\alpha_1 + \beta\beta_1)v_1 + (\alpha\alpha_2 + \beta\beta_2)v_2 + \cdots + (\alpha\alpha_n + \beta\beta_n)v_n
\end{aligned}$$

belongs to W. Hence W is a subspace of V(Lemma 40.3). [Notice that $v_1, v_2, \ldots, v_n$ are not assumed to be distinct.] $\square$

We extend this theorem to infinite subsets of V.

40.8 Theorem: *Let V be a vector space over a field K and let $A = \{v_i : i \in I\}$ be a (finite or infinite) nonempty subset of V. Then the set*

$$W = \{\alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \cdots + \alpha_n v_{i_n} \in V : \alpha_1, \alpha_2, \ldots, \alpha_n \in K,\ v_{i_1}, v_{i_2}, \ldots, v_{i_n} \in A,\ n \in \mathbb{N}\}$$

of all finite linear combinations of the vectors in A is a subspace of V.

Proof: Since A is not empty, $W \neq \varnothing$. If $\alpha, \beta \in K$ and $u, w \in W$, then

$$u = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \cdots + \alpha_n v_{i_n}, \quad w = \beta_1 v_{j_1} + \beta_2 v_{j_2} + \cdots + \beta_m v_{j_m}$$

with suitable $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\beta_1, \beta_2, \ldots, \beta_n \in K$, $v_{i_1}, v_{i_2}, \ldots, v_{i_n}, v_{j_1}, v_{j_2}, \ldots, v_{j_m} \in A$, $n, m \in \mathbb{N}$ and

$$\begin{aligned}\alpha u + \beta w &= \alpha(\alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \cdots + \alpha_n v_{i_n}) + \beta(\beta_1 v_{j_1} + \beta_2 v_{j_2} + \cdots + \beta_m v_{j_m})\\ &= \alpha\alpha_1 v_{i_1} + \alpha\alpha_2 v_{i_2} + \cdots + \alpha\alpha_n v_{i_n} + \beta\beta_1 v_{j_1} + \beta\beta_2 v_{j_2} + \cdots + \beta\beta_m v_{j_m}\end{aligned}$$

is a K-linear combination of the vectors $v_{i_1}, v_{i_2}, \ldots, v_{i_n}, v_{j_1}, v_{j_2}, \ldots, v_{j_m}$ in A. So $\alpha u + \beta w$ belongs to W and W is a subspace of V. $\square$

40.9 Definition: The subspace W of Theorem 40.7 or Theorem 40.8 is called the *K-span of A*, or the *K-span of the vectors in A*, or the *subspace spanned by the (vectors in) A*. It will be denoted by $s_K(A)$. In case $A = \{v_1, v_2, \ldots, v_n\}$ is a finite set, we write $s_K(v_1, v_2, \ldots, v_n)$ instead of $s_K(\{v_1, v_2, \ldots, v_n\})$. By convention, we put $s_K(\varnothing) = \{0\}$. When there is no need to refer to the field K of scalars, we speak of the *span* of A, and denote it by $s(A)$.

The next lemma justifies the convention $s_K(\varnothing) = \{0\}$.

40.10 Lemma: *Let V be a vector space over a field K and let $A \subseteq V$. Then $s_K(A)$ is the smallest subspace of V which contains A. More exactly, if U is a subspace of V and $A \subseteq U$, then $s_K(A) \subseteq U$.*

Proof: If $A = \varnothing$, then $s_K(A) = \{0\} \subseteq U$ for any subspace U of V and the theorem is proved in this case. Suppose now $A \neq \varnothing$. If $A \subseteq U$ and U is a subspace of V, then every linear combination of the vectors in A belongs to U by Lemma 40.5. Hence $s(A) \subseteq U$. □

40.11 Lemma: *Let V be a vector space over a field K and let A,B be subspaces of V such that $A \subseteq s(B)$ and $B \subseteq s(A)$. Then $s(A) = s(B)$.*

Proof: Since $A \subseteq s(B)$ and $s(B)$ is a subspace of V, we have $s(A) \subseteq s(B)$ by Lemma 40.10. In like manner, since $B \subseteq s(A)$ and $s(A)$ is a subspace of V, we get $s(B) \subseteq s(A)$. Thus $s(A) = s(B)$. □

40.12 Examples: (a) Let V be a vector space over a field K and let A be a subset of V having only one element, say $A = \{v\}$. Then the span $s(v)$ of A is the set

$$\{\alpha v \in V : \alpha \in K\}$$

of all scalar multiples of v. In case $K = \mathbb{R}$ and $V = \mathbb{R}^2$ or $V = \mathbb{R}^3$, this span is usually identified with the line through the origin determined by v.

(b) Let V be a vector space over a field K and let u,v be two vectors in V. The span $s(u,v)$ of these vectors is

$$\{\alpha u + \beta v \in V : \alpha,\beta \in K\}.$$

In case v is a scalar multiple γu of u, we have

$$s(u,v) = \{\alpha u + \beta v \in V : \alpha,\beta \in K\} = \{(\alpha + \beta\gamma)u : \alpha,\beta \in K\} = \{\delta u : \delta \in K\} = s(u).$$

We see it is possible that $A \subset B$ and $s(A) = s(B)$. In case $K = \mathbb{R}$ and $V = \mathbb{R}^3$ and v is not a scalar multiple of u, this span is usually identified with the plane through the origin determined by u and v.

(c) In the vector space $\mathbb{R}^2$ over $\mathbb{R}$, consider the set

$$A = \{(1,0),(2,0),\dots,(10\,000,0)\}.$$

The span $s(A)$ is easily seen to be $\{(a,0) \in \mathbb{R}^2 : a \in \mathbb{R}\}$, which is also the span of $\{(1,0)\} \subseteq \mathbb{R}^2$. Thus the number of vectors in A may be large, but this does not imply that $s(A)$ is a "big" subspace.

(d) Let V be a vector space over a field K and let A,B be subsets of V with $A \subseteq B$. Then $A \subseteq B \subseteq s(B)$ and, since $s(B)$ is a subspace of V, Lemma 40.10 yields $s(A) \subseteq s(B)$. So $A \subseteq B$ implies $s(A) \subseteq s(B)$. We have seen in Example 40.12(b) and Example 40.12(c) that $A \subset B$ does not necessarily imply $s(A) \subset s(B)$.

Exercises

1. Let V be a vector space over a field K. If W is a subspace of V and U is a subspace of W, prove that U is a subspace of V.

2. Prove that the set of all sequences of real numbers converging to 0 is a subspace of the $\mathbb{R}$-vector space S (see § 39, Ex. 5). What do you say about the set of all convergent sequences, all bounded sequences, all monotonic sequences, and all sequences with at most finitely many nonzero terms?

3. Consider the $\mathbb{R}$-vector space V of Example 39.2(c). Determine whether the following are subspaces of V: the set of bounded functions, the set of even functions, the set of integrable functions, the set of monotonic functions, the set of functions with at most finitely many points of discontinuity (all with domains [0,1]).

4. Determine whether
$$\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 : 5\alpha - 4\beta + 2\gamma = 0\}$$
$$\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 : 5\alpha - 4\beta + 2\gamma \geqslant 0\}$$
$$\{(\alpha,\beta,\gamma) \in {\mathbb{Z}_{11}}^3 : 5\alpha - 4\beta + 2\gamma = 0\}$$
$$\{(\alpha,\beta,\gamma) \in \mathbb{C}^3 : 5\alpha - 4\beta + 2\gamma \geqslant 0\}$$
are subspaces of the vector spaces indicated.

5. Is $(1,0,1) \in \mathbb{R}^3$ in the $\mathbb{R}$-span of $\{(5,4,1), (3,2,2)\} \subseteq \mathbb{R}^3$?