§41

Factor Spaces

In the preceding paragraph, we discussed subspaces, which are the analogues of subgroups and subrings. We now wish to discuss the analogues of factor groups and factor rings.

Let V be a vector space over a field K and let W be a subspace of V. Then W is a subgroup of the additive group V, and we can build the factor group V/W. The elements of V/W are cosets $v + W$, where $v \in V$; the sum of two cosets $v_1 + W$ and $v_2 + W$ is the coset $(v_1 + v_2) + W$. The operation on V/W is denoted by "+", but "+" designates in V/W an operation distinct from the addition in V. The question arises: is it possible to define on V/W a kind of multiplication by scalars so that V/W becomes a vector space over K? The most natural multiplication $\circ$ is to put

$$\alpha \circ (v + W) = \alpha v + W \qquad \text{for all } \alpha \in K,\ v + W \in V/W.$$

We prove that $\circ$ is well defined. To this end, we must show that the implication

$$u + W = v + W \quad \Rightarrow \quad \alpha u + W = \alpha v + W \qquad (\text{for all } \alpha \in K,\ v,u \in V)$$

is valid. This implication is equivalent to

$$u - v \in W \quad \Rightarrow \quad \alpha u + W = \alpha v + W$$

hence to

$$u - v \in W \quad \Rightarrow \quad \alpha u - \alpha v \in W.$$

Since W is a subspace of V, it is closed under multiplication by scalars, hence $\alpha(u - v) \in W$ whenever $u - v \in W$. This proves that the above multiplication $\circ$ by scalars is well defined.

It is now quite straightforward to show that $(V/W,+,K,\circ)$ is a vector space. For any $\alpha,\beta \in K$, $u,v \in V$, we have

$$\begin{aligned} (1) \qquad \alpha \circ \big((u + W) + (v + W)\big) &= \alpha \circ \big((u + v) + W\big) \\ &= \alpha(u + v) + W \end{aligned}$$

$$
\begin{aligned}
&= (\alpha u + \alpha v) + W\\
&= (\alpha u + W) + (\alpha v + W)\\
&= \alpha \circ (u + W) \;+\; \alpha \circ (v + W),\\
\text{(2)}\qquad\qquad (\alpha + \beta) \circ (u + W) &= (\alpha + \beta) u + W\\
&= (\alpha u + \beta u) + W\\
&= (\alpha u + W) + (\beta u + W)\\
&= \alpha \circ (u + W) \;+\; \beta \circ (u + W),\\
\text{(3)}\qquad\qquad (\alpha\beta) \circ (u + W) &= (\alpha\beta) u + W\\
&= \alpha(\beta u) + W\\
&= \alpha \circ (\beta u + W)\\
&= \alpha \circ \big(\beta \circ (u + W)\big),\\
\text{(4)}\qquad\qquad 1 \circ (u + W) &= 1u + W\\
&= u + W.
\end{aligned}
$$

Thus $(V/W,+,K,\circ)$ is a vector space.

We employed the symbol "$\circ$" chiefly to emphasize that multiplication of the elements in V/W by scalars is distinct from the multiplication of the elements in V by scalars. For ease of notation, we shall drop "$\circ$" and write simply $\alpha(u + W)$ instead of $\alpha \circ (u + W)$. Also, we will write V/W for $(V/W,+,K,\circ)$. The following theorem summarizes this discussion.

41.1 Theorem: *Let V be a vector space over a field K and let W be a subspace of V. Then the abelian group V/W is a vector space over K if multiplication by scalars is defined by*

$$\alpha(u + W) = \alpha u + W \qquad \textit{for all } \alpha \in K,\ u \in V. \qquad \square$$

41.2 Definition: Let V be a vector space over a field K and let W be a subspace of V. The K-vector space V/W of Theorem 41.1 is called the *factor space of V by W*, or the *factor space V mod(ulo) W.*

We know that factor groups (rings) are closely related to homomorphisms of groups (rings). The same is true for factor spaces.

41.3 Definition: Let V and U be vector spaces over the same field K. A mapping $\varphi: V \to U$ is called a *vector space homomorphism,* or a *K-linear transformation,* or a *K-linear mapping* if

$$(v_1 + v_2)\varphi = v_1\varphi + v_2\varphi \quad \text{and} \quad (\alpha v)\varphi = \alpha(v\varphi)$$

for all $v_1,v_2,v \in V$, $\alpha \in K$. When there is no need to emphasize the field of scalars, we speak simply of linear transformations or linear mappings.

More exactly, when $(V,+,K,\cdot)$ and $(U,\oplus,K,\circ)$ are vector spaces, the mapping $\varphi: V \to U$ is a vector space homomorphism provided

$$(v_1 + v_2)\varphi = v_1\varphi \oplus v_2\varphi \quad \text{and} \quad (\alpha v)\varphi = \alpha \circ (v\varphi)$$

for all $v_1,v_2,v \in V$, $\alpha \in K$. Notice that the field of scalars of both vector spaces are the same. A linear transformation from V into U cannot be defined if V and U are vector spaces over different fields.

A mapping $\varphi: V \to U$ such that $(v_1 + v_2)\varphi = v_1\varphi + v_2\varphi$ for all $v_1,v_2 \in V$ is said to be *additive*. So an additive mapping is just a group homomorphism from the group $(V,+)$ into $(U,+)$. A mapping $\varphi: V \to U$ such that $(\alpha v)\varphi = \alpha(v\varphi)$ for all $v \in V$, $\alpha \in K$ is said to be *homogeneous*. A homogeneous mapping is one that preserves the multiplication by scalars. A mapping may be additive without being homogeneous, and it may be homogeneous without being additive. In order to be a linear transformation, a mapping shoud be both additive and homogeneous.

A vector space homomorphism is therefore a homomorphism of additive groups which preserves multiplication by scalars as well. This observation enables us to use the properties of group homomorphisms whenever we investigate vector space homomorphisms.

41.4 Lemma: *Let V and U be a vector spacer over a field K. A function $\varphi: V \to U$ is a K-linear mapping if and only if*

$$(\alpha v_1 + \beta v_2)\varphi = \alpha(v_1\varphi) + \beta(v_2\varphi)$$

for all $\alpha,\beta \in K$, $v_1,v_2 \in V$.

Proof: If φ is a K-linear mapping and $\alpha,\beta \in K$, $v_1,v_2 \in V$, then

$$(\alpha v_1 + \beta v_2)\varphi = (\alpha v_1)\varphi + (\beta v_2)\varphi = \alpha(v_1\varphi) + \beta(v_2\varphi)$$

since φ is additive and homogeneous. Conversely, if we have $(\alpha v_1 + \beta v_2)\varphi = \alpha(v_1\varphi) + \beta(v_2\varphi)$ for all $\alpha,\beta \in K$, $v_1,v_2 \in V$, then, choosing $\alpha = \beta = 1$, we see that φ is additive and choosing $\beta = 0$, we see that φ is homogeneous. □

41.5 Lemma: *Let V,U be a vector spacer over a field K and let* $\varphi\colon V \to U$ *be a vector space homomorphism.*

(1) $0\varphi = 0$.

(2) $(-v)\varphi = -(v\varphi)$ *for all* $v \in V$.

(3) $(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)\varphi = \alpha_1(v_1\varphi) + \alpha_2(v_2\varphi) + \cdots + \alpha_n(v_n\varphi)$ *for all* $\alpha_1,\alpha_2,\dots,\alpha_n \in K$ *and for all* $v_1,v_2,\dots,v_n \in V$.

(4) $(nv)\varphi = n(v\varphi)$ *for all* $n \in \mathbb{Z}$.

Proof: (1),(2),(4) follow respectively from (1),(2),(4) of Lemma 20.3 and (3) follows from Lemma 20.3(3) by the homogeneity of φ, or from Lemma 41.4 by induction on n. □

41.6 Examples: (a) Let K be a field and let $\varphi\colon K^3 \longrightarrow K^2$. Then
$$(\lambda,\mu,\nu) \to (\lambda,\mu)$$

$$\begin{aligned}\big(\alpha(\lambda,\mu,\nu) + \beta(\lambda',\mu',\nu')\big)\varphi &= \big((\alpha\lambda,\alpha\mu,\alpha\nu) + (\beta\lambda',\beta\mu',\beta\nu')\big)\varphi\\ &= \big((\alpha\lambda + \beta\lambda', \alpha\mu + \beta\mu', \alpha\nu + \beta\nu')\big)\varphi\\ &= (\alpha\lambda + \beta\lambda', \alpha\mu + \beta\mu')\\ &= (\alpha\lambda,\alpha\mu) + (\beta\lambda',\beta\mu')\\ &= \alpha(\lambda,\mu) + \beta(\lambda',\mu')\\ &= \alpha\big((\lambda,\mu,\nu)\big)\varphi + \beta\big((\lambda',\mu',\nu')\big)\varphi\end{aligned}$$

for all $\alpha,\beta \in K$, (λ,μ,ν), $(\lambda',\mu',\nu') \in K^3$. Hence φ is a K-linear transformation.

(b) Let K be a field and let $\varphi\colon K^2 \longrightarrow K^2$. Then
$$(\lambda,\mu) \to (\mu,\lambda)$$

$$\begin{aligned}\big(\alpha(\lambda,\mu) + \beta(\lambda',\mu')\big)\varphi &= \big((\alpha\lambda,\alpha\mu) + (\beta\lambda',\beta\mu')\big)\varphi = \big((\alpha\lambda + \beta\lambda', \alpha\mu + \beta\mu')\big)\varphi\\ &= (\alpha\mu + \beta\mu', \alpha\lambda + \beta\lambda') = (\alpha\mu,\alpha\lambda) + (\beta\mu',\beta\lambda') = \alpha(\mu,\lambda) + \beta(\mu',\lambda')\\ &= \alpha\big((\lambda,\mu)\big)\varphi + \beta\big((\lambda',\mu')\big)\varphi\end{aligned}$$

for all $\alpha,\beta \in K$, $(\lambda,\mu),(\lambda',\mu') \in K^2$. Hence φ is a vector space homomorphism.

(c) The mapping $\varphi\colon \boldsymbol{C}^1([0,1]) \to \mathbb{R}$ is $\mathbb{R}$-linear, because

$$f \;\to\; f(\tfrac{1}{2})$$

$(\alpha f + \beta g)\varphi = (\alpha f + \beta g)(\frac{1}{2}) = (\alpha f)(\frac{1}{2}) + (\beta g)(\frac{1}{2}) = \alpha\big(f(\frac{1}{2})\big) + \beta\big(g(\frac{1}{2})\big) = \alpha(f\varphi) + \beta(g\varphi)$

for all $\alpha,\beta \in \mathbb{R}$, $f,g \in \boldsymbol{C}^1([0,1])$. Likewise, for any $\gamma \in [0,1]$, the mapping

$$\begin{aligned} \varphi_\gamma\colon \boldsymbol{C}^1([0,1]) &\to \mathbb{R} \\ f \quad &\to \quad f(\gamma) \end{aligned}$$

is a vector space homomorphism.

(d) Let V,U be vector spaces over a field K and let W be a subspace of V. If $\varphi\colon V \to U$ is a vector space homomorphism, then its restriction

$$\varphi_W\colon W \to U$$

to W is also a vector space homomorphism, because

$$(\alpha w_1 + \beta w_2)\varphi = \alpha(w_1\varphi) + \beta(w_2\varphi)$$

for all $\alpha,\beta \in K$, $w_1,w_2 \in W$, as this holds in fact for all $\alpha,\beta \in K$, $w_1,w_2 \in V$ (Lemma 41.4).

(e) Let V,U be vector spaces over a field K and let K_1 be a field contained in K. Then V,U are vector spaces over K_1, too (Example 39.2(f)). If $\varphi\colon V \to U$ is a K-linear mapping, then φ is also a K_1-linear mapping, because $(\alpha w_1 + \beta w_2)\varphi = \alpha(w_1\varphi) + \beta(w_2\varphi)$ for all $\alpha,\beta \in K_1$, $v_1,v_2 \in V$, as this holds in fact for all $\alpha,\beta \in K$, $v_1,v_2 \in V$ (Lemma 41.4).

(f) The mapping $T\colon \boldsymbol{C}^2([0,1]) \to \boldsymbol{C}([0,1])$ is a vector space homomorphism

$$y \to \; y'' - 5y' + 6y$$

because

$$\begin{aligned} (\alpha y_1 + \beta y_2)T &= (\alpha y_1 + \beta y_2)'' - 5(\alpha y_1 + \beta y_2)' + 6(\alpha y_1 + \beta y_2) \\ &= \alpha y_1'' + \beta y_2'' - 5(\alpha y_1' + \beta y_2') + 6(\alpha y_1 + \beta y_2) \\ &= \alpha(y_1'' - 5y_1' + 6y_1) + \beta(y_2'' - 5y_2' + 6y_2) \\ &= \alpha(y_1T) + \beta(y_2T) \end{aligned}$$

for any $\alpha,\beta \in \mathbb{R}$, $y_1,y_2 \in \boldsymbol{C}^2([0,1])$. In the theory of ordinary differential equations, this mapping is called a *linear differential operator* and is usually denoted by $D^2 - 5D + 6$.

In the rest of this paragraph, we establish the counterparts of certain theorems discussed in §§ 20, 21.

41.7 Theorem: *Let* V,U,W *be a vector spaces over a field* K. *Let* $\varphi: V \to U$ *and* $\psi: U \to W$ *be vector space homomorphisms. Then the composition mapping* $\varphi\psi: V \to W$ *is a vector space homomorphism from* V *into* W.

Proof: $\varphi\psi$ is a group homomorphism (is additive) by Theorem 20.4. Also

$$(\alpha v)\varphi\psi = ((\alpha v)\varphi)\psi = (\alpha(v\varphi))\psi = \alpha((v\varphi)\psi) = \alpha(v(\varphi\psi))$$

for all $\alpha \in K$, $v \in V$, hence $\varphi\psi$ is homogeneous. Thus $\varphi\psi$ is a vector space homomorphism. □

41.8 Theorem: *Let* V,U *be a vector spaces over a field* K *and let* $\varphi: V \to U$ *be* K*-linear. Then* $Im\ \varphi = \{v\varphi \in U: v \in V\}$ *is a subspace of* U *and* $Ker\ \varphi = \{v \in V: v\varphi = 0\}$ *is a subspace of* V.

Proof: *Im* φ is a subgroup of $(U,+)$ by Theorem 20.6. Also, if $u \in Im\ \varphi$ and $\alpha \in K$, then $u = v\varphi$ for some $v \in V$, so $\alpha u = \alpha(v\varphi) = (\alpha v)\varphi$, so $\alpha u \in Im\ \varphi$. Thus *Im* φ is closed under multiplication by scalars. Therefore *Im* φ is a subspace of U.

Ker φ is a subgroup of $(V,+)$ by Theorem 20.6. Also, if $v \in Ker\ \varphi$ and $\alpha \in K$, then $v\varphi = 0$, so $(\alpha v)\varphi = \alpha(v\varphi) = \alpha 0 = 0$ by Lemma 39.4(6), so $\alpha v \in Ker\ \varphi$. Thus *Ker* φ is closed under multiplication by scalars. Therefore *Ker* φ is a subspace of V. □

41.9 Definition: Let V,U be a vector spaces over a field K. A vector space homomorphism $\varphi: V \to U$ is called a *vector space isomorphism* if φ is one-to-one and onto. If there is a vector space isomorphism from V onto U, we say *V is isomorphic to U*, and write $V \cong U$.

So a vector space isomorphism is an additive group isomorphism which preserves multiplication by scalars. We use the same symbol "$\cong$" for isomorphic vector spaces as for isomorphic groups. This will not lead to confusion. When there is any danger of confusion, we will state explicitly whether we mean vector space isomorphism or group isomorphism.

41.10 Lemma: *Let V,U,W be a vector spaces over a field K and let $\varphi\colon V \to U$ and $\varphi\colon U \to W$ be vector space isomorphisms.*
(1) *The composition $\varphi\psi\colon V \to W$ is a vector space isomorphism from V onto W.*
(2) *The inverse $\varphi^{-1}\colon U \to V$ of φ is a vector space isomorphism from U onto V.*

Proof: (1) $\varphi\psi$ is a vector space homomorphism by Theorem 41.7, and $\varphi\psi$ is one-to-one and onto by Theorem 3.13. So $\varphi\psi$ is a vector space isomorphism.

(2) $\varphi^{-1}\colon U \to V$ is an isomorphism of additive groups by Lemma 20.11(2). We have only to show that φ^{-1} preserves multiplication by scalars. Let $u \in U$ and $\alpha \in K$. Then $u = v\varphi$ for some uniquely determined $v \in V$, namely $v = u\varphi^{-1}$. Since $(\alpha v)\varphi = \alpha(v\varphi) = \alpha u$, we have $(\alpha u)\varphi^{-1} = \alpha v$. Thus $(\alpha u)\varphi^{-1} = \alpha v = \alpha(u\varphi^{-1})$. So φ^{-1} preserves multiplication by scalars. □

As in the case of groups, we see that

$$V \cong V,$$
$$\text{if } V \cong U, \text{ then } U \cong V,$$
$$\text{if } V \cong U \text{ and } U \cong W, \text{ then } V \cong W$$

for all K-vector spaces V,U,W, where K is any field. Thus $\cong$ is an equivalence relation, but me must refrain from saying "on the *set* of K-vector spaces".

In view of the symmetry property of $\cong$, it is legitimate to say that V and U are isomorphic when V is isomorphic to U.

41.11 Theorem: *Let V be a vector space over a field K and let W be a subspace of V. Then the mapping*

$$\begin{aligned} \nu\colon V &\to V/W \\ v &\to v + W \end{aligned}$$

is a vector space homomorphism. It is onto V/W. Also, $\mathrm{Ker}\,\nu = W$. (This mapping ν is called the *natural* or *canonical homomorphism from V onto V/W*).

Proof: ν is an additive group homomorphism from V onto V/W such that $Ker\ \nu = W$ (Theorem 20.12). Since $(\alpha v)\nu = \alpha v + W = \alpha(v + W) = \alpha(v\nu)$ for all $\alpha \in K$, $v \in V$, we see that ν is a vector space homomorphism. □

41.12 Theorem (Fundamental theorem on homomorphisms): *Let K be a field. Let V, V_1 be K-vector spaces and let $\varphi: V \to V_1$ be a vector space homomorphism. Let $W = Ker\ \varphi$ and let $\nu: V \to V/W$ be the associated natural homomorphism.*
Then there is a vector space homomorphism $\psi: V/W \to V_1$ such that

$$\nu\varphi = \psi.$$

Proof: From Theorem 20.15, we know that $\psi: V/W \to V_1$ is a well defined,

$$v + W \to v\varphi$$

one-to-one homomorphism of additive groups with $\nu\varphi = \psi$. For all $\alpha \in K$, $v \in V$, we have $\big(\alpha(v + W)\big)\psi = (\alpha v + W)\psi = (\alpha v)\varphi = \alpha(v\varphi) = \alpha\big((v + W)\psi\big)$, so ψ is homogeneous and is therefore a vector space homomorphism. □

41.13 Theorem: *Let V, U be vector spaces over a field K and let $\varphi: V \to U$ be a vector space homomorphism. Then*

$$V/Ker\ \varphi \cong Im\ \varphi \quad (\textit{as vector spaces}).$$

Proof: From Theorem 20.16 and its proof, we know that

$$\psi: V/Ker\ \varphi \to Im\ \varphi$$
$$v + Ker\ \varphi \to v\varphi$$

is an isomorphism of additive groups, thus $V/Ker\ \varphi \cong Im\ \varphi$ as groups; and ψ is a vector space homomorphism by Theorem 41.12. Hence $\psi: V/Ker\ \varphi \to Im\ \varphi$ is a vector space isomorphism and $V/Ker\ \varphi \cong Im\ \varphi$ as vector spaces. □

41.14 Theorem: *Let V, V_1 be vector spaces over a field K and let $\varphi: V \to V_1$ be a vector space homomorphism from* V ***onto*** *V_1.*
(1) *Each subspace W of V with $Ker\ \varphi \subseteq W$ is mapped to a subspace of V_1, which will be denoted by W_1.*

(2) *If W,U are subspaces of V with $Ker\ \varphi \subseteq W \subseteq U$, then $W_1 \subseteq U_1$.*

(3) *If W,U are subspaces of V with $Ker\ \varphi \subseteq W$ and $Ker\ \varphi \subseteq U$, and if $W_1 \subseteq U_1$, then $W \subseteq U$.*

(4) *If W,U are subspaces of V with $Ker\ \varphi \subseteq W$ and $Ker\ \varphi \subseteq U$, and if $W_1 = U_1$, then $W = U$.*

(5) *If S is any subspace of V_1, then there is a subspace W of V such that $Ker\ \varphi \subseteq W$ and $W_1 = S$.*

(6) *If U is a subspace of V with $Ker\ \varphi \subseteq U$, then $V/U \cong V_1/U_1$.*

Proof: (1) For each subspace W of V with $Ker\ \varphi \subseteq W$, we put $W_1 = Im\ \varphi_W$, as in Theorem 21.1. Then W_1 is a subspace of V_1 by Theorem 41.8. and Example 41.6(d).

(2),(3),(4) These follow from parts (2),(3),(4) of Theorem 21.1 on regarding the subspaces merely as additive subgroups.

(5) From Theorem 21.1(5) and its proof, we know that $W := \{w \in V\colon w\varphi \in S\}$ is a subgroup of $(V,+)$ with $Ker\ \varphi \subseteq W$ and $W_1 = S$. For any $\alpha \in K$ and $w \in W$, we have $w\varphi \in S$, so $\alpha(w\varphi) \in S$, so $(\alpha w)\varphi \in S$, so $\alpha w \in W$ and W is in fact a subspace of V.

(6) Let $\nu'\colon V_1 \to V_1/U_1$ be the natural homomorphism. Then ν' and $\varphi\nu'\colon V \to V_1 \to V_1/U_1$ are vector space homomorphisms (Theorem 41.11, Theorem 41.7) with $Ker\ \varphi\nu' = U$ and $Im\ \varphi\nu' = V_1/U_1$ (Theorem 21.1(6),(7)). Hence, by Theorem 41.13, we have the vector space isomorphism

$$V/Ker\ \varphi\nu' \cong Im\ \varphi\nu'$$
$$V/U \cong V_1/U_1.$$

□

41.15 Theorem: *Let V be a vector space over a field K and let W be a subspace of V. The subspaces of V/W are given by U/W, where U runs through the subspaces of V containing W. In other words, for each subspace X of V/W, there is a unique subspace U of V such that $W \subseteq U$ and $X = U/W$. When X_1 and X_2 are subspaces of V/W, say with $X_1 = U_1/W$ and $X_2 = U_2/W$, where U_1, U_2 are subspaces of V containing W, then $X_1 \subseteq X_2$ if and only if $U_1 \subseteq U_2$. Furthermore, there holds*

$$V/W \,\big/\, U/W \cong V/U \qquad (\textit{vector space isomorphism}).$$

Proof: The natural homomorphism $\nu\colon V \to V/W$ is onto by Theorem 41.11. We may therefore apply Theorem 41.14. This theorem states that any subspace of V/W is of the form $Im\, \nu_U$ for some subspace U of V with $Ker\, \nu \subseteq U$. Now

$$\begin{aligned} Im\, \nu_U &= \{u\nu \in V/W\colon u \in U\} \\ &= \{u + W \in V/W\colon u \in U\} = U/W \end{aligned}$$

and $Ker\, \nu = W$ by Theorem 41.11. Thus the subspaces of V/W are given by U/W, where U's are subspaces of V containing W. By Theorem 41.14(2),(3),(4), $U_1/W \subseteq U_2/W$ if and only if $U_1 \subseteq U_2$, and $U_1/W \neq U_2/W$ whenever $U_1 \neq U_2$. Finally, by Theorem 41.14(6)

$$V/U \cong Im\, \nu_V/Im\, \nu_U = V/W \,/\, U/W \quad \text{as vector spaces.} \qquad \square$$

41.16 Theorem: *Let V be a vector space over a field K and let U,W be subspaces of V. Then $U \cap W$ and $U + W$ are subspaces of V and*

$$W/U \cap W \;\cong\; U + W\,/U \qquad (\textit{vector space isomorphism}).$$

Proof: $U \cap W$ is a subspace of V by Example 40.6(g). Also, $U + W$ is a subgroup of $(V,+)$ by Lemma 19.4 and, for any $\alpha \in K$, $v \in U + W$, there are $u \in U$ and $w \in W$ with $v = u + w$, so that

$$\alpha v = \alpha(u + w) = \alpha u + \alpha w \in U + W$$

since $\alpha u \in U$ and $\alpha w \in W$; and so $U + W$ is closed under multiplication by scalars and $U + W$ is a subspace of V.

We consider the restriction

$$\nu_W\colon W \to V/U$$

to W of the natural homomorphism $\nu\colon V \to V/U$. By Theorem 41.11, ν is a vector space homomorphism; by Example 41.6(d), ν_W is a vector space homomorphism, so

$$W/Ker\, \nu_W \cong Im\, \nu_W \qquad \text{(as vector spaces)}$$

according to Theorem 41.13. From the proof of Theorem 21.3, we know that $Ker\, \nu_W = U \cap W$ and $Im\, \nu_W = U + W\,/U$, as may also be established directly. Hence

$$W/U \cap W \;\cong\; U + W\,/U. \qquad \square$$

Exercises

1. Let V be a vector space over a field K and let W be a subgroup of the additive group $(V,+)$. For all α in K and for all $v + W$ in the factor group V/W, we write $\alpha \circ (v + W) = \alpha v + W$. Prove that $(\alpha, v + W) \to \alpha \circ (v + W)$ is a well defined mapping from $K \times (V/W)$ into V/W if and only if W is a subspace of V.

2. (cf. §20, Ex 14) Let $\varphi\colon V \to V_1$ be a vector space homomorphism, let W be a subspace of V such that $W \leqslant Ker\ \varphi$, and let $\nu\colon V \to V/W$ be the associated natural homomorphism. Show that there is a vector space homomorphism $\psi\colon V/W \to V_1$ such that $\nu\psi = \varphi$ and $Ker\ \psi = (Ker\ \varphi)/W$. What happens when we drop the condition $W \leqslant Ker\ \varphi$?