

§44 Determinants

With each (square) matrix over a field K , we associate an element of K , called the determinant of the matrix. In this paragraph, we study the properties of determinants.

Determinants arise in many contexts. For example, if a_1, a_2, b_1, b_2 elements of a field K and if the equations

$$\begin{aligned} a_1x + a_2y &= 0 \\ b_1x + b_2y &= 0 \end{aligned}$$

hold, then $(a_1b_2 - a_2b_1)x = (a_1b_2 - a_2b_1)y = 0$. In §17, we called $a_1b_2 - a_2b_1$ the determinant of the matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$.

Now let a_3, b_3, c_1, c_2, c_3 be further elements of K . If

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ b_1x + b_2y + b_3z &= 0 \\ c_1x + c_2y + c_3z &= 0, \end{aligned}$$

then, multiplying the first equation by $b_2c_3 - b_3c_2$, the second by $a_3c_2 - a_2c_3$, the third by $a_2b_3 - a_3b_2$ and adding them, we get $Dx = 0$, where

$$D = a_1b_2c_3 - a_1b_3c_2 + a_3b_2c_1 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1.$$

One obtains also $Dy = 0$ and $Dz = 0$. Here D is a sum of $6 = 3!$ terms $\mp a_i b_j c_k$, where $\{i, j, k\} = \{1, 2, 3\}$ and the sign of $a_i b_j c_k$ is + or - according as $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$ is an even or odd permutation in S_3 .

Similarly, when we try to eliminate x, y, z, u from the equations

$$\begin{aligned} a_1x + a_2y + a_3z + a_4u &= 0 \\ b_1x + b_2y + b_3z + b_4u &= 0 \\ c_1x + c_2y + c_3z + c_4u &= 0 \\ d_1x + d_2y + d_3z + d_4u &= 0, \end{aligned}$$

we get $D'x = D'y = D'z = D'u = 0$, where D is a sum of $24 = 4!$ terms $\mp a_i b_j c_k d_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and the sign of $a_i b_j c_k d_l$ is + or - according as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$ is an even or odd permutation in S_4 .

This pattern continues. The expressions we get in this way are called determinants. On changing a to α_1 , b to α_2 , c to α_3 , etc., the formal definition reads as follows.

44.1 Definition: Let K be a field and

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nm} \end{pmatrix} = (\alpha_{ij})$$

be an $n \times n$ square matrix with entries from K . Then the element

$$\sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma}$$

of K is called the *determinant of the matrix A* . It will be denoted as

$$\det A, \quad \text{or } \det(A), \quad \text{or } \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nm} \end{vmatrix}, \quad \text{or } |\alpha_{ij}|.$$

Hence $\det A$ is a sum of $n!$ terms. These summands are obtained from the product $\alpha_{11} \alpha_{22} \cdots \alpha_{nn}$ of the entries in the main diagonal by permuting the second indices in all the $n!$ ways and attaching a "+" or "-" sign according as the permutation is even or odd. Each summand, aside from its sign, is the product of n entries of the matrix, the entries being from distinct rows and distinct columns. The determinant can also be written

$$\sum_{\sigma \in A_n} \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma} - \sum_{\sigma \in S_n \setminus A_n} \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma}.$$

44.2 Remarks: (1) Determinants are defined for square matrices only. Nonsquare matrices do not have a determinant. Note that the determinant of the 1×1 matrix (α) is equal to $\alpha \in K$.

(2) Definition 44.1 makes sense when K is merely a commutative ring. The theory in this paragraph extends immediately to the case where K is a commutative ring with identity. We will not need this general theory.

We observe only: when R is a subring of K , and all entries of $A \in \text{Mat}_n(K)$ are in R , then $\det A$ is in fact an element of R .

Some fundamental properties of determinants are collected in the next lemmas.

44.3 Lemma: *Let K be a field and $A \in \text{Mat}_n(K)$. Then $\det A = \det A^t$. (The determinant does not change when rows are changed to columns.)*

Proof: Let $A = (\alpha_{ij})$ and $A^t = (\beta_{ij})$, so that $\alpha_{ij} = \beta_{ji}$ for all i, j . Then

$$\det A = \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma}.$$

As σ runs through S_n , so does σ^{-1} . Hence

$$\det A = \sum_{\sigma \in S_n} \mathbb{E}(\sigma^{-1}) \alpha_{1,1\sigma^{-1}} \alpha_{2,2\sigma^{-1}} \cdots \alpha_{n,n\sigma^{-1}}.$$

Using commutativity of multiplication in K , we reorder the factors in each summand with regard to their second indices and get

$$\det A = \sum_{\sigma \in S_n} \mathbb{E}(\sigma^{-1}) \alpha_{1\sigma,1} \alpha_{2\sigma,2} \cdots \alpha_{n\sigma,n}.$$

Since $\mathbb{E}(\sigma^{-1}) = \mathbb{E}(\sigma)$ for all $\sigma \in S_n$ (in case $n \geq 2$; if $n = 1$, there is nothing to prove, for then $A = A^t$),

$$\begin{aligned} \det A &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1\sigma,1} \alpha_{2\sigma,2} \cdots \alpha_{n\sigma,n} \\ &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \beta_{1,1\sigma} \beta_{2,2\sigma} \cdots \beta_{n,n\sigma} \\ &= \det A^t. \end{aligned}$$

□

44.4 Lemma: *Let K be a field and $A \in \text{Mat}_n(K)$. If each element in a particular row (column) of A is multiplied by $\gamma \in K$, then the determinant of the new matrix thus obtained is equal to $\gamma \det A$.*

Proof: In view of Lemma 44.3, it suffices to prove the statement about rows only. Let $A = (\alpha_{ij})$. Assume that the elements of the k -th row are multiplied by γ . The new matrix is (β_{ij}) , where $\beta_{ij} = \alpha_{ij}$ for $i \neq k$ and $\beta_{kj} = \gamma\alpha_{kj}$. Thus

$$\begin{aligned} |\beta_{ij}| &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \beta_{1,1\sigma} \cdots \beta_{k,k\sigma} \cdots \beta_{n,n\sigma} \\ &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \cdots (\gamma\alpha_{k,k\sigma}) \cdots \alpha_{n,n\sigma} \\ &= \gamma \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \cdots \alpha_{k,k\sigma} \cdots \alpha_{n,n\sigma} \\ &= \gamma |\alpha_{ij}|. \quad \square \end{aligned}$$

44.5 Lemma: Let K be a field and $A \in \text{Mat}_n(K)$. Assume that each element α_{kj} in the k -th row of A is a sum $\beta_{kj} + \gamma_{kj}$ (each element α_{kj} in the k -th column of A is a sum $\beta_{ik} + \gamma_{ik}$). Then $\det A$ is a sum of two determinants:

$$\det A = \det B + \det C,$$

where B resp. C is identical with A , except for the k -th row (column), in which α_{kj} are replaced by β_{kj} resp. γ_{kj} (α_{ik} are replaced by β_{ik} resp. γ_{ik}). Symbolically

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{k1} + \gamma_{k1} & \beta_{k2} + \gamma_{k2} & \cdots & \beta_{kn} + \gamma_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{k1} & \beta_{k2} & \cdots & \beta_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{k1} & \gamma_{k2} & \cdots & \gamma_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix}$$

and

$$\begin{vmatrix} \alpha_{11} & \cdots & \beta_{1k} + \gamma_{1k} & \cdots & \alpha_{1n} \\ \alpha_{21} & \cdots & \beta_{2k} + \gamma_{2k} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \beta_{nk} + \gamma_{nk} & \cdots & \alpha_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_{11} & \cdots & \beta_{1k} & \cdots & \alpha_{1n} \\ \alpha_{21} & \cdots & \beta_{2k} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \beta_{nk} & \cdots & \alpha_{nn} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \cdots & \gamma_{1k} & \cdots & \alpha_{1n} \\ \alpha_{21} & \cdots & \gamma_{2k} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \gamma_{nk} & \cdots & \alpha_{nn} \end{vmatrix}.$$

Proof: The proof is shorter than the wording of the lemma. It will be sufficient to prove the assertion involving rows only, and this follows from summing

$$\begin{aligned} & \mathbb{E}(\sigma)\alpha_{1,1\sigma}\cdots\alpha_{k,k\sigma}\cdots\alpha_{n,n\sigma} \\ &= \mathbb{E}(\sigma)\alpha_{1,1\sigma}\cdots(\beta_{k,k\sigma} + \gamma_{k,k\sigma})\cdots\alpha_{n,n\sigma} \\ &= \mathbb{E}(\sigma)\alpha_{1,1\sigma}\cdots\beta_{k,k\sigma}\cdots\alpha_{n,n\sigma} + \mathbb{E}(\sigma)\alpha_{1,1\sigma}\cdots\gamma_{k,k\sigma}\cdots\alpha_{n,n\sigma} \end{aligned}$$

over all $\sigma \in S_n$. □

The last two lemmas mean that the determinant of a matrix is a linear function of any one of its rows or columns.

44.6 Lemma: *Let K be a field, $A \in \text{Mat}_n(K)$ and $\gamma \in K$. Then $\det(\gamma A) = \gamma^n \cdot \det A$.*

Proof: This follows from n successive applications of Lemma 44.4. Alternatively, observe that, when we put $A = (\alpha_{ij})$, each summand $\mathbb{E}(\sigma)(\gamma\alpha_{1,1\sigma})(\gamma\alpha_{2,2\sigma})\cdots(\gamma\alpha_{n,n\sigma})$ of $\det(\gamma A)$ is γ^n times a summand $\mathbb{E}(\sigma)\alpha_{1,1\sigma}\alpha_{2,2\sigma}\cdots\alpha_{n,n\sigma}$ of $\det A$, and conversely. □

44.7 Lemma: *Let K be a field and $A, B \in \text{Mat}_n(K)$. If B is obtained from A by interchanging two rows (columns) of A , then $\det B = -\det A$ (the determinant changes sign when two rows (columns) are interchanged.)*

Proof: We prove the statement about rows only. Assume $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, and assume that B is obtained from A by interchanging the k -th and m -th rows of A so that $\beta_{ij} = \alpha_{ij}$ for all i, j with $i \neq k, i \neq m$ and $\beta_{kj} = \alpha_{mj}$, $\beta_{mj} = \alpha_{kj}$ for all j . Then

$$\det B = \sum_{\sigma \in S_n} \mathbb{E}(\sigma)\beta_{1,1\sigma}\cdots\beta_{k,k\sigma}\cdots\beta_{m,m\sigma}\cdots\beta_{n,n\sigma}$$

As σ ranges over S_n , so does $(km)\sigma$. Hence we have

$$\begin{aligned}
 \det B &= \sum_{\sigma \in S_n} \mathbb{E}((km)\sigma) \beta_{1,1(km)\sigma} \cdots \beta_{k,k(km)\sigma} \cdots \beta_{m,m(km)\sigma} \cdots \beta_{n,n(km)\sigma} \\
 &= - \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \beta_{1,1\sigma} \cdots \beta_{k,m\sigma} \cdots \beta_{m,k\sigma} \cdots \beta_{n,n\sigma} \\
 &= - \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \cdots \alpha_{m,m\sigma} \cdots \alpha_{k,k\sigma} \cdots \alpha_{n,n\sigma} \\
 &= - \det A. \quad \square
 \end{aligned}$$

44.8 Lemma: Let K be a field and $A, B \in \text{Mat}_n(K)$, $n \geq 2$. If B is obtained from A by a permutation τ of the rows (columns) of A , then $\det B = \mathbb{E}(\tau) \det A$.

Proof: Let $A = (\alpha_{ij})$ and $B = (\beta_{ij})$. We give a proof of the assertion about rows only. The hypothesis is that $\beta_{ij} = \alpha_{i\tau(j)}$ for some τ in S_n . We write τ as a product of transpositions:

$$\tau = \tau_1 \tau_2 \cdots \tau_s \quad (\tau_1, \tau_2, \dots, \tau_s \text{ are transpositions in } S_n)$$

so that $\mathbb{E}(\tau) = (-1)^s$ by definition. We introduce matrices

$$A = A_0, A_1, A_2, \dots, A_{s-1}, A_s = B,$$

where each A_r is obtained from A_{r-1} ($r = 1, 2, \dots, s$) by interchanging two rows:

$$A_0 = (\alpha_{ij}), A_1 = (\alpha_{i\tau_1(j)}), A_2 = (\alpha_{i\tau_1\tau_2(j)}), \dots, A_{s-1} = (\alpha_{i\tau_1\tau_2\cdots\tau_{s-1}(j)}), A_s = (\alpha_{i\tau_1\tau_2\cdots\tau_{s-1}\tau_s(j)}).$$

Then, using Lemma 44.7 repeatedly,

$$\begin{aligned}
 \det B = \det A_s &= - \det A_{s-1} = (-1)^2 \det A_{s-2} = (-1)^3 \det A_{s-3} \\
 &= \cdots = (-1)^s \det A_0 = \mathbb{E}(\tau) \det A. \quad \square
 \end{aligned}$$

44.9 Lemma: Let K be a field and $A \in \text{Mat}_n(K)$, $n \geq 2$. If two rows (columns) of A are identical, then $\det A = 0$.

Proof: One usually argues as follows. Interchanging the two identical rows (columns), the $\det A$ does not change. But it becomes $-\det A$ by Lemma 44.7. Hence $\det A = -\det A$. Thus $2\det A = 0$. One concludes from this that $\det A = 0$.

This conclusion is justified when we can divide by 2 in K , that is to say, if the multiplicative inverse of 2 exists in K . Let us recall that 2 is an abbreviation of $1_K + 1_K$, where 1_K is the identity of K . Since any nonzero element of K has an inverse in K , the conclusion is valid when K is a field in which $1_K + 1_K \neq 0$. If, however, $1_K + 1_K = 0$ (as in \mathbb{Z}_2), this argument does not work.

We give an argument which works irrespective of whether $1_K + 1_K = 0$ or not. We prove the statement about rows only. Reordering the rows of A by a suitable permutation in S_n , we obtain a matrix B in which the first two rows are identical and $\det B = \mathbb{E}(\tau)\det A$. Since $\det B = 0$ if and only if $\det A = 0$, we may assume, without loss of generality, that the first and second rows of A are identical. We prove $\det A = 0$ under this assumption.

Let $A = (\alpha_{ij})$, with $\alpha_{1j} = \alpha_{2j}$ for all j . If $n = 2$, then $\det A = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{11} & \alpha_{12} \end{vmatrix} = \alpha_{11}\alpha_{12} - \alpha_{12}\alpha_{11} = 0$. Let us suppose now $n \geq 3$. Then

$$\det A = \sum_{\sigma \in A_n} \alpha_{1,1\sigma} \alpha_{2,2\sigma} \alpha_{3,3\sigma} \cdots \alpha_{n,n\sigma} - \sum_{\sigma \in S_n \setminus A_n} \alpha_{1,1\sigma} \alpha_{2,2\sigma} \alpha_{3,3\sigma} \cdots \alpha_{n,n\sigma}. \quad (i)$$

As σ runs through A_n , the permutation $(12)\sigma$ runs through $S_n \setminus A_n$. Hence

the subtrahend $\sum_{\sigma \in S_n \setminus A_n} \alpha_{1,1\sigma} \alpha_{2,2\sigma} \alpha_{3,3\sigma} \cdots \alpha_{n,n\sigma}$ in (i) is equal to

$$\begin{aligned} &= \sum_{\sigma \in A_n} \alpha_{1,1(12)\sigma} \alpha_{2,2(12)\sigma} \alpha_{3,3(12)\sigma} \cdots \alpha_{n,n(12)\sigma} \\ &= \sum_{\sigma \in A_n} \alpha_{1,2\sigma} \alpha_{2,1\sigma} \alpha_{3,3\sigma} \cdots \alpha_{n,n\sigma} \\ &= \sum_{\sigma \in A_n} \alpha_{2,1\sigma} \alpha_{1,2\sigma} \alpha_{3,3\sigma} \cdots \alpha_{n,n\sigma} \text{ (commutativity of multiplication)} \\ &= \sum_{\sigma \in A_n} \alpha_{1,1\sigma} \alpha_{2,2\sigma} \alpha_{3,3\sigma} \cdots \alpha_{n,n\sigma} \text{ (first two rows are identical),} \end{aligned}$$

which is the minuend in (i). Hence $\det A = 0$. □

It will be convenient to identify the i -th row

$$\alpha_{i1} \alpha_{i2} \cdots \alpha_{in}$$

of a matrix $A = (\alpha_{ij}) \in \text{Mat}_n(K)$, where K is a field, with the vector

$$(\alpha_{i1} \ \alpha_{i2} \ \cdots \ \alpha_{in})$$

in $K^n = \text{Mat}_{1 \times n}(K)$. Similarly, the j -th column

$$\alpha_{1j}$$

$$\alpha_{2j}$$

$$\vdots$$

$$\alpha_{nj}$$

of A will be identified with the vector (matrix)

$$\begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$$

in $\text{Mat}_{n \times 1}(K)$. Thus it is meaningful to speak of K -linear (in)dependence of rows and columns of a matrix. Likewise, we can add two rows (columns) and multiply them by scalars.

44.10 Lemma: *Let K be a field and $A, B \in \text{Mat}_n(K)$, $n \geq 2$. Suppose that B is obtained from A by multiplying a particular row (column) of A by some $\gamma \in K$ and adding it to a different row (column) of A . Then $\det B = \det A$. (The determinant does not change when we add a multiple of a row (column) to another.)*

Proof: We prove the assertion about rows only. Suppose that the k -th row in A is multiplied by $\gamma \in K$ and added to the m -th row. Writing $A = (\alpha_{ij})$, $B = (\beta_{ij})$, we have $\beta_{mj} = \gamma\alpha_{kj} + \alpha_{mj}$ and $\beta_{ij} = \alpha_{ij}$ for $i \neq m$. Lemma 44.5 gives $\det B = \det C + \det A$, where $C \in \text{Mat}_n(K)$ is identical with A except for the m -th row, which is γ times the k -th row of A . By Lemma 44.4, $\det C = \gamma \det D$, where $D \in \text{Mat}_n(K)$ is identical with A except that the m -th row of $D =$ the k -th row of $A =$ the k -th row of D . Then $\det D = 0$ by Lemma 44.9 and $\det B = \gamma \det D + \det A = \det A$. \square

44.11 Lemma: *Let K be a field and $A \in \text{Mat}_n(K)$. If every entry in a particular row (column) of A is equal to $0 \in K$, then $\det A = 0$.*

Proof: Let $A = (\alpha_{ij})$. Under the hypothesis of the lemma, each summand $\mathbb{E}(\sigma)\alpha_{1,1\sigma}\alpha_{2,2\sigma}\cdots\alpha_{n,n\sigma}$ ($\sigma \in S_n$) of $\det A$ is zero, for one of the factors is zero. Hence $\det A = 0$. \square

44.12 Lemma: *Let K be a field and $A \in \text{Mat}_n(K)$. If the rows (columns) of A are linearly dependent over K , then $\det A = 0$.*

Proof: When $n = 1$, A must be the matrix (0) (see Example 42.2(a)), and $\det A = 0$. Assume now $n \geq 2$. We prove the assertion about rows only. If the rows of A are K -linearly dependent, then there are $\beta_1, \beta_2, \dots, \beta_n$ in K such that

$$\beta_1(\text{1st row}) + \beta_2(\text{2nd row}) + \dots + \beta_n(\text{n-th row}) = (0, 0, \dots, 0)$$

and not all of $\beta_1, \beta_2, \dots, \beta_n$ are equal to $0 \in K$. Suppose $\beta_k \neq 0$. Then β_k has an inverse β_k^{-1} in K . We multiply the i -th row by $\beta_i\beta_k^{-1}$ and add it to the k -th row; we do this for each $i \neq k$. Then we obtain a matrix B whose determinant is equal to $\det A$ by Lemma 44.10. On the other hand, the k -th row of B consists entirely of zeroes and $\det B = 0$ by Lemma 44.11. Hence $\det A = 0$. \square

Now we want to discuss the calculation of determinants. In practice, determinants are almost never computed from the definition

$$\sum_{\sigma \in S_n} \mathbb{E}(\sigma)\alpha_{1,1\sigma}\alpha_{2,2\sigma}\cdots\alpha_{n,n\sigma}$$

Rather, a determinant of an $n \times n$ matrix is expressed in terms of the determinants of certain $(n-1) \times (n-1)$ matrices, these in turn in terms of the determinants of certain $(n-2) \times (n-2)$ matrices and so on, until we come to 2×2 matrices, whose determinants are evaluated readily. This reduction process is known as the expansion of a determinant along (or by) a row (column). To describe this process, we introduce a definition.

44.13 Definition: Let K be a field and $A = (\alpha_{ij}) \in \text{Mat}_n(K)$, with $n \geq 2$. Let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i -th row and the j -th column of A , which intersect at the entry α_{ij} of A . Then

$(-1)^{i+j} \det M_{ij}$ is called the *cofactor* of α_{ij} in A . We write A_{ij} for the cofactor of α_{ij} in A .

The following lemma justifies the terminology.

44.14 Lemma: *Let K be a field and $A = (\alpha_{ij}) \in \text{Mat}_n(K)$, where $n \geq 2$. Let k, m be fixed elements of $\{1, 2, \dots, n\}$. Collecting together all terms containing α_{km} in*

$$\det A = \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma}$$

we write

$$\det A = \alpha_{km} c_{km} + \text{terms not containing } \alpha_{km}$$

The c_{km} having been defined uniquely in this way, we claim:

- (1) $c_{mm} = \text{cofactor of } \alpha_{mm} = A_{mm}$,
- (2) $c_{mm} = A_{mm}$ for any $m = 1, 2, \dots, n$,
- (3) $c_{km} = A_{km}$ for any $k, m = 1, 2, \dots, n$.

Proof: (1) We have

$$\begin{aligned} \det A &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma} \\ &= \sum_{\substack{\sigma \in S_n \\ n\sigma = n}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n-1,(n-1)\sigma} \alpha_{n,n\sigma} + \sum_{\substack{\sigma \in S_n \\ n\sigma \neq n}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \\ &\quad \alpha_{n,n\sigma} \\ &= \alpha_{mm} \sum_{\substack{\sigma \in S_n \\ n\sigma = n}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n-1,(n-1)\sigma} + \text{terms not involving } \alpha_{mm}. \end{aligned}$$

Any $\sigma \in S_n$ with $n\sigma = n$ can be regarded as a permutation in S_{n-1} , and any permutation in S_{n-1} can be regarded as a permutation in S_n with $n\sigma = n$. Here $\mathbb{E}(\sigma)$ is independent of whether we regard σ as an element of S_n or of S_{n-1} . Hence

$$c_{mm} = \sum_{\substack{\sigma \in S_n \\ n\sigma = n}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n-1,(n-1)\sigma} = \sum_{\sigma \in S_{n-1}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n-1,(n-1)\sigma}$$

$$= \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,n-1} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \cdots & \alpha_{n-1,n-1} \end{vmatrix} = A_{nm}$$

(2) We prove $c_{nm} = A_{nm}$ for all $m = 1, 2, \dots, n$. The case $m = n$ having been settled in part (1) above, we assume $m < n$. Consider the matrix $B =$

$$\begin{matrix} \alpha_{11} & \cdots & \alpha_{1,m-1} & \alpha_{1n} & \alpha_{1,m+1} & \cdots & \alpha_{1,n-1} & \alpha_{1m} \\ \alpha_{21} & \cdots & \alpha_{2,m-1} & \alpha_{2n} & \alpha_{2,m+1} & \cdots & \alpha_{2,n-1} & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1,1} & \cdots & \alpha_{n-1,m-1} & \alpha_{n-1,n} & \alpha_{n-1,m+1} & \cdots & \alpha_{n-1,n-1} & \alpha_{n-1,m} \\ \alpha_{n1} & \cdots & \alpha_{n,m-1} & \alpha_{nm} & \alpha_{n,m+1} & \cdots & \alpha_{n,n-1} & \alpha_{nm} \end{matrix}$$

obtained from A by interchanging the m -th and n -th columns. Then we have $\det A = -\det B$ by Lemma 44.7 and, by part (1),

$$\det B = \alpha_{nm} \det M + \text{terms not involving } \alpha_{nm} \tag{ii}$$

where M is the $(n-1) \times (n-1)$ matrix we obtain from B by deleting its n -th row and n -th column. A glance at B reveals that M is obtained from

$$M_{nm} = \begin{matrix} \alpha_{11} & \cdots & \alpha_{1,m-1} & \alpha_{1,m+1} & \cdots & \alpha_{1,n-1} & \alpha_{1n} \\ \alpha_{21} & \cdots & \alpha_{2,m-1} & \alpha_{2,m+1} & \cdots & \alpha_{2,n-1} & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1,1} & \cdots & \alpha_{n-1,m-1} & \alpha_{n-1,m+1} & \cdots & \alpha_{n-1,n-1} & \alpha_{n-1,n} \end{matrix}$$

by $n - 1 - m$ interchanges of columns. Hence $\det M = (-1)^{n-1-m} \det M_{nm} = -(-1)^{n+m} \det M_{nm} = -A_{nm}$. Substituting this in (ii), we get

$$\begin{aligned} \det A = -\det B &= \alpha_{nm} (-\det M) - \text{terms not involving } \alpha_{nm} \\ &= \alpha_{nm} A_{nm} + \text{terms not involving } \alpha_{nm} \end{aligned}$$

as was to be shown.

(3) We now prove $c_{km} = A_{km}$ for all k, m . The case $k = n$ having been settled in part (2), we assume $k < n$. We consider the matrix C obtained from B by interchanging the k -th and n -th rows. Then $\det C = -\det B = \det A$ by Lemma 44.7 and, by part (1),

$$\det C = \alpha_{km} \det N + \text{terms not involving } \alpha_{km}$$

where N is the $(n-1) \times (n-1)$ matrix we obtain from C by deleting its n -th row and n -th column. The matrix N is obtained from M_{kn} by $n - m - 1$ interchanges of columns and $n - k - 1$ interchanges of rows. Hence

$$\det N = (-1)^{(n-m-1)+(n-k-1)} \det M_{kn} = (-1)^{k+m} \det M_{kn} = A_{kn}$$

and $\det A = \det C = \alpha_{kn} A_{kn} + \text{terms not involving } \alpha_{kn}$

This completes the proof. \square

44.15 Theorem: Let K be a field, $A = (\alpha_{ij}) \in \text{Mat}_n(K)$, where $n \geq 2$. Let A_{ij} be the cofactor of α_{ij} in A . Then

$$\det A = \alpha_{i1} A_{i1} + \alpha_{i2} A_{i2} + \cdots + \alpha_{in} A_{in}$$

$$\det A = \alpha_{1j} A_{1j} + \alpha_{2j} A_{2j} + \cdots + \alpha_{nj} A_{nj}$$

for all i, j .

Proof: We have $\det A = \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma}$

$$= \sum_{\substack{\sigma \in S_n \\ i\sigma=1}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma} + \sum_{\substack{\sigma \in S_n \\ i\sigma=2}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma} + \cdots$$

$$+ \sum_{\substack{\sigma \in S_n \\ i\sigma=n}} \mathbb{E}(\sigma) \alpha_{1,1\sigma} \alpha_{2,2\sigma} \cdots \alpha_{n,n\sigma}$$

$$= \alpha_{i1}^c A_{i1} + \alpha_{i2}^c A_{i2} + \cdots + \alpha_{in}^c A_{in}$$

$$= \alpha_{i1} A_{i1} + \alpha_{i2} A_{i2} + \cdots + \alpha_{in} A_{in}$$

for any i . This proves the first formula. Applying it with A^t , j in place of A , i , we obtain the second formula. \square

The first formula in Theorem 44.15 is known as the *expansion of det A along the i-th row*, the second as the *expansion of det A along the j-th column*. Each element in the i -th row (j -th column) contributes a term, more specifically α_{ij} contributes $\alpha_{ij} A_{ij}$, where A_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column of α_{ij} times 1 or -1, determined by the chessboard pattern

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The expansion along a row or column is sometimes given as a recursive definition of determinants in terms of determinants of smaller size.

A specific determinant is computed as follows. If a row or column consists of zeroes, the determinant is 0. Otherwise, we choose and fix a row or column. It will be convenient to choose the row (or column) which has the largest number of zeroes. At least one of the entries on the fixed row (column), say β , is distinct from 0. If a column (row) intersects our fixed row (column) at the entry γ , we add $-\gamma\beta^{-1}$ times the column (row) of β to that column (row). We do this for each column (row). This does not change the determinant, but our fixed row (column) will consist entirely of zeroes, except for the entry β . Expanding the determinant along the fixed row (column), we see that the determinant is equal to βD , where D is the new cofactor of β . We repeat the same procedure with the determinant D , and obtain $D = \beta' D'$, say. Then we repeat the same process with D' , etc., until we come to a 2×2 or 3×3 determinant which can be computed easily.

44.16 Examples: (a) Let K be a field and x_1, x_2, \dots, x_n elements in K . We

$$\text{evaluate } \det(x_i^{j-1}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

This is known as the Vandermonde determinant. Let us denote it by D_n . We add $-x_n$ times the i -th row to the $(i + 1)$ -st row ($i = 1, 2, \dots, n - 1$). The only nonzero entry in the new last column will be the entry 1 in the 1st row, n -th column. Expanding D_n along the last column, and taking the factor $x_i - x_n$ in the i -th column of the cofactor outside the determinant sign by Lemma 44.4 ($i = 1, 2, \dots, n-1$), we obtain

$$D_n = (-1)^{n-1} (x_1 - x_n)(x_2 - x_n) \cdots (x_{n-1} - x_n) D_{n-1}.$$

This holds for any n . Thus

$$\begin{aligned}
 D_n &= (-1)^{n-1}(x_1 - x_n)(x_2 - x_n) \cdots (x_{n-1} - x_n) \times \\
 &\quad (-1)^{n-2}(x_1 - x_{n-1})(x_2 - x_{n-1}) \cdots (x_{n-2} - x_{n-1})D_{n-2} \\
 &= \cdots \\
 &= (-1)^{(n-1)+(n-2)+\cdots+1} (x_1 - x_n)(x_2 - x_n) \cdots (x_{n-3} - x_n)(x_{n-2} - x_n)(x_{n-1} - \\
 &\quad x_n) \\
 &\quad (x_1 - x_{n-1})(x_2 - x_{n-1}) \cdots (x_{n-3} - x_{n-1})(x_{n-2} - x_{n-1}) \\
 &\quad (x_1 - x_{n-2})(x_2 - x_{n-2}) \cdots (x_{n-3} - x_{n-2}) \\
 &\quad \cdots \cdots \\
 &\quad (x_1 - x_2).
 \end{aligned}$$

Changing the sign of the $\binom{n}{2}$ factors on the right hand side and noting that $(n-1) + (n-2) + \cdots + 1 = \binom{n}{2}$, we finally get

$$D_n = \prod_{i>j} (x_i - x_j),$$

the product being over all $\binom{n}{2}$ pairs (i,j) , where $i,j = 1,2, \dots, n$ and $i > j$.

(b) Let K be a field. The determinant of a matrix $(\alpha_{ij}) \in Mat_n(K)$, where $\alpha_{ij} = 0$ whenever $i > j$, which may be written symbolically

$$\begin{vmatrix}
 \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\
 & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\
 & & \alpha_{33} & \cdots & \alpha_{3n} \\
 \mathbf{0} & & & \ddots & \\
 & & & & \alpha_{nn}
 \end{vmatrix}$$

can be evaluated by expanding successively along the first columns. One finds immediately that $|\alpha_{ij}| = \alpha_{11}\alpha_{22}\alpha_{33} \cdots \alpha_{nn}$. Likewise, the determinant

$$\begin{vmatrix}
 \alpha_{11} & & & \\
 \alpha_{21} & \alpha_{22} & & \\
 \alpha_{31} & \alpha_{32} & \alpha_{33} & \\
 & & & \mathbf{0}
 \end{vmatrix}$$

To express these equations more succinctly, we introduce a definition.

44.18 Definition: Let K be a field and $A = (\alpha_{ij}) \in \text{Mat}_n(K)$, where $n \geq 2$. The $n \times n$ matrix obtained from A by replacing the entry α_{ij} by the cofactor A_{ij} of α_{ij} in A is called the *adjoint* of A . Hence
the adjoint of $(\alpha_{ij}) = (A_{ij})$.

Using this terminology, the equations above can be written as matrix equations,

$$\begin{aligned} A \cdot (\text{adjoint of } A)^t &= (\det A)I \\ A^t \cdot (\text{adjoint of } A) &= (\det A)I. \end{aligned}$$

Taking the transposes of both sides in the second equation, we obtain

44.19 Theorem: Let K be a field and $A \in \text{Mat}_n(K)$, where $n \geq 2$. Then

$$A \cdot (\text{adjoint of } A)^t = (\det A)I = (\text{adjoint of } A)^t \cdot A. \quad \square$$

44.20 Theorem: Let K be a field and $A \in \text{Mat}_n(K)$. Then A is invertible if and only if $\det A \in K^\times$. If this is the case, the inverse A^{-1} of A is given by the formula

$$A^{-1} = \frac{1}{\det A} (\text{adjoint of } A)^t,$$

where $\frac{1}{\det A}$ denotes the inverse of $\det A$ in K .

Proof: If $\det A = 0$, then $(\det A)I = 0 \in \text{Mat}_n(K)$, hence, by Theorem 44.19, A is a left zero divisor and a right zero divisor in the ring $\text{Mat}_n(K)$. From Lemma 29.10, we deduce that A cannot have a left or right inverse.

Otherwise, $\det A \neq 0$ and $\det A$ has an inverse $\frac{1}{\det A}$ in K . If $n = 1$, then $A = (\det A)$ and $(\frac{1}{\det A})$ is the inverse of A . If $n \geq 2$, we multiply the members of the equations in Theorem 44.19 by $\frac{1}{\det A}$ and obtain

$$A \cdot \frac{1}{\det A} (\text{adjoint of } A)^t = I = \frac{1}{\det A} (\text{adjoint of } A)^t \cdot A.$$

This shows that $\frac{1}{\det A} (\text{adjoint of } A)^t$ is an inverse of A . So $A \in GL(n, K)$

and, since $GL(n, K)$ is a group, A has a unique inverse. Hence

$$\frac{1}{\det A} (\text{adjoint of } A)^t \text{ is the inverse } A^{-1} \text{ of } A. \quad \square$$

The next theorem is another testimony for the use of determinants.

44.21 Theorem: *Let K be a field and $A \in Mat_n(K)$. Then $\det A = 0$ if and only if the rows (columns) of A are linearly dependent over K .*

Proof: If the rows (columns) of A are linearly dependent over K , then $\det A = 0$ by Lemma 44.12.

Assume conversely that $\det A = 0$. Let $A = (\alpha_{ij})$. Let V be an n -dimensional K -vector space and let $\{v_1, v_2, \dots, v_n\}$ be a K -basis of V . Then the K -linear transformation $T \in L_K(V, V)$, given by

$$v_i T = \sum_{j=1}^n \alpha_{ij} v_j$$

has the associated matrix $(\alpha_{ij}) = A$, which is *not* invertible since $\det A = 0$. So A is not a unit in $Mat_n(K)$ and T is not a unit in $L_K(V, V)$. Thus T is not an isomorphism. From Theorem 42.22, we conclude that T is not one-to-one. Thus $\text{Ker } T \neq \{0\}$. Let $v \in \text{Ker } T$, $v \neq 0$. We have

$$v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$$

for some suitable scalars $\beta_j \in K$. Here not all of β_j are equal to 0, because $v \neq 0$ and $\{v_1, v_2, \dots, v_n\}$ is a K -basis of V . Then

$$0 = vT = \left(\sum_{i=1}^n \beta_i v_i \right) T = \sum_{i=1}^n \beta_i (v_i T) = \sum_{i=1}^n \beta_i \sum_{j=1}^n \alpha_{ij} v_j = \sum_{j=1}^n \left(\sum_{i=1}^n \beta_i \alpha_{ij} \right) v_j,$$

so
$$\sum_{i=1}^n \beta_i \alpha_{ij} = 0 \quad \text{for } j = 1, 2, \dots, n$$

since $\{v_1, v_2, \dots, v_n\}$ is a K -basis of V . Thus

$$\beta_1(\text{1st row}) + \beta_2(\text{2nd row}) + \dots + \beta_n(\text{n-th row}) = (0, 0, \dots, 0)$$

with scalars $\beta_1, \beta_2, \dots, \beta_n \in K$ which are not all equal to 0. So the rows of A are K -linearly dependent. Repeating the same argument with A^t , we see that the columns of A , too, are K -linearly dependent. \square

We now establish the multiplication rule for determinants.

44.21 Theorem: *Let K be a field, $n \in \mathbb{N}$.*

(1) $\det(AB) = (\det A)(\det B)$ for all $A, B \in \text{Mat}_n(K)$.

(2) $\det I = 1$.

(3) $\det A^{-1} = (\det A)^{-1}$ for all $A \in \text{GL}(n, K)$.

Proof: (2) That $\det I = 1$ is a special case of the formula for the determinant of a diagonal matrix discussed in Example 44.16(b). And (3) follows from (1) and (2): $(\det A^{-1})(\det A) = \det(A^{-1}A) = \det I = 1$.

We prove (1). Let $A = (\alpha_{ij})$, $B = (\beta_{ij})$, $AB = (\gamma_{ij})$, so that $\gamma_{ij} = \sum_{k=1}^n \alpha_{ik}\beta_{kj}$ for all

$$\begin{aligned}
 i, j. \text{ Then } \det(AB) &= \begin{vmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{vmatrix} \\
 &= \begin{vmatrix} \sum_{k_1=1}^n \alpha_{1k_1} \beta_{k_1 1} & \sum_{k_2=1}^n \alpha_{1k_2} \beta_{k_2 2} & \cdots & \sum_{k_n=1}^n \alpha_{1k_n} \beta_{k_n n} \\ \sum_{k_1=1}^n \alpha_{2k_1} \beta_{k_1 1} & \sum_{k_2=1}^n \alpha_{2k_2} \beta_{k_2 2} & \cdots & \sum_{k_n=1}^n \alpha_{2k_n} \beta_{k_n n} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{k_1=1}^n \alpha_{nk_1} \beta_{k_1 1} & \sum_{k_2=1}^n \alpha_{nk_2} \beta_{k_2 2} & \cdots & \sum_{k_n=1}^n \alpha_{nk_n} \beta_{k_n n} \end{vmatrix} \\
 &= \sum_{k_1=1}^n \sum_{k_2=1}^n \cdots \sum_{k_n=1}^n \begin{vmatrix} \alpha_{1k_1} \beta_{k_1 1} & \alpha_{1k_2} \beta_{k_2 2} & \cdots & \alpha_{1k_n} \beta_{k_n n} \\ \alpha_{2k_1} \beta_{k_1 1} & \alpha_{2k_2} \beta_{k_2 2} & \cdots & \alpha_{2k_n} \beta_{k_n n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{nk_1} \beta_{k_1 1} & \alpha_{nk_2} \beta_{k_2 2} & \cdots & \alpha_{nk_n} \beta_{k_n n} \end{vmatrix} \quad (\text{Lemma 44.5}) \\
 &= \sum_{k_1=1}^n \sum_{k_2=1}^n \cdots \sum_{k_n=1}^n \beta_{k_1 1} \beta_{k_2 2} \cdots \beta_{k_n n} \begin{vmatrix} \alpha_{1k_1} & \alpha_{1k_2} & \cdots & \alpha_{1k_n} \\ \alpha_{2k_1} & \alpha_{2k_2} & \cdots & \alpha_{2k_n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{nk_1} & \alpha_{nk_2} & \cdots & \alpha_{nk_n} \end{vmatrix} \quad (\text{Lemma 44.4}).
 \end{aligned}$$

In this n -fold sum, k_1, k_2, \dots, k_n run independently over $1, 2, \dots, n$. If, however, any two of k_1, k_2, \dots, k_n are equal, then the determinant $|\alpha_{ik_j}|$ in the n -fold sum has two identical columns and therefore vanishes (Lemma 44.9). So we may disregard those combinations of the indices k_1, k_2, \dots, k_n which contain two equal values, and restrict the n -fold summation to those combinations of k_1, k_2, \dots, k_n such that k_1, k_2, \dots, k_n are all distinct. Then the n -fold sum becomes

$$\begin{aligned}
 & \sum_{\substack{1 \ 2 \ \dots \ n \\ (k_1 \ k_2 \ \dots \ k_n) \in S_n}} \beta_{k_1 1} \beta_{k_2 2} \cdots \beta_{k_n n} \begin{vmatrix} \alpha_{1k_1} & \alpha_{1k_2} & \cdots & \alpha_{1k_n} \\ \alpha_{2k_1} & \alpha_{2k_2} & \cdots & \alpha_{2k_n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{nk_1} & \alpha_{nk_2} & \cdots & \alpha_{nk_n} \end{vmatrix} \\
 &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \beta_{1\sigma,1} \beta_{2\sigma,2} \cdots \beta_{n\sigma,n} \cdot \mathbb{E}(\sigma) \begin{vmatrix} \alpha_{1,1\sigma} & \alpha_{1,2\sigma} & \cdots & \alpha_{1,n\sigma} \\ \alpha_{2,1\sigma} & \alpha_{2,2\sigma} & \cdots & \alpha_{2,n\sigma} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n,1\sigma} & \alpha_{n,2\sigma} & \cdots & \alpha_{n,n\sigma} \end{vmatrix} \\
 &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \beta_{1\sigma,1} \beta_{2\sigma,2} \cdots \beta_{n\sigma,n} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} \quad (\text{Lemma 44.8}) \\
 &= \sum_{\sigma \in S_n} \mathbb{E}(\sigma) \beta_{1\sigma,1} \beta_{2\sigma,2} \cdots \beta_{n\sigma,n} (\det A) = (\det A)(\det B) \quad (\text{Lemma 44.3}).
 \end{aligned}$$

Hence $\det AB = (\det A)(\det B)$. □

The equation $\det AB = (\det A)(\det B)$ may also be written in the forms

$$\begin{aligned}
 \det AB &= (\det A)(\det B^t), \\
 \det AB &= (\det A^t)(\det B), \\
 \det AB &= (\det A^t)(\det B^t).
 \end{aligned}$$

So there are four versions of the multiplication rule for determinants, known as the rows by columns multiplication, rows by rows multiplication, columns by columns multiplication, column by rows multiplication, which are respectively described below:

If K is a field, $(\alpha_{ij}), (\beta_{ij}), (\gamma_{ij}) \in Mat_{n \times m}(K)$, and if

$$\gamma_{ij} = \sum_{k=1}^n \alpha_{ik} \beta_{kj} \quad \text{for all } i, j, \quad \text{or}$$

$$\gamma_{ij} = \sum_{k=1}^n \alpha_{ik} \beta_{jk} \quad \text{for all } i, j, \quad \text{or}$$

$$\gamma_{ij} = \sum_{k=1}^n \alpha_{kt} \beta_{kj} \quad \text{for all } i, j, \quad \text{or}$$

$$\gamma_{ij} = \sum_{k=1}^n \alpha_{kt} \beta_{jk} \quad \text{for all } i, j,$$

then $|\gamma_{ij}| = |\alpha_{ij}| |\beta_{ij}|$.

Restricting the mapping $det: Mat_n(K) \rightarrow K$ to

$$GL(n, K) = \{A \in Mat_n(K) : det A \in K^\times\}$$

(Theorem 44.20), we obtain a group homomorphism

$$det: GL(n, K) \rightarrow K^\times.$$

The kernel

$$\{A \in Mat_n(K) : det A = 1\}$$

of this determinant homomorphism is a normal subgroup of $GL(n, K)$, known as the *special linear group of degree n over K* , and denoted as $SL(n, K)$.

Exercises

1. Verify that the determinant of a 3×3 matrix (α_{ij}) can be computed as follows. We write the first column of the matrix to the right of the matrix and the second column to the right of the last written copy of the first column:

$$\begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{11} & \alpha_{12} & \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{21} & \alpha_{22} & \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{31} & \alpha_{32} & \end{array}$$

We take the products of the upper-left, lower-right diagonals (full lines) unchanged, the products of the lower-right, upper-left diagonals (broken lines) with a minus sign. The sum of these six products is the determi-

nant of (α_{ij}) . (This rule cannot be extended to $n \times n$ matrices if n is greater than 3).

2. Compute the determinants of the following matrices over \mathbb{Q} :

$$\begin{aligned} \text{(a)} & \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}; & \text{(b)} & \begin{pmatrix} 1 & 4 & 5 \\ -1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}; & \text{(c)} & \begin{pmatrix} 1 & 2 & 5 \\ -1 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}; & \text{(d)} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & -4 & 2 \end{pmatrix}; \\ \text{(e)} & \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 3 & 2 \end{pmatrix}; & \text{(f)} & \begin{pmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & -3 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 4 \end{pmatrix}; & \text{(g)} & \begin{pmatrix} 1 & 0 & 0 & 7 & -3 \\ 4 & 4 & 0 & 1 & 0 \\ 2 & -8 & 2 & -1 & 4 \\ 2 & 1 & 5 & -2 & -1 \\ -1 & 0 & 1 & 0 & 2 \end{pmatrix}. \end{aligned}$$

3. Find $\det A$ if A is the matrix $\begin{pmatrix} \bar{2} & \bar{1} & \bar{1} \\ \bar{3} & \bar{1} & \bar{0} \\ \bar{0} & \bar{3} & \bar{2} \end{pmatrix}$ in $Mat_3(\mathbb{Z}_7)$.

4. Expand along the third column:

$$\begin{vmatrix} 1 & 0 & 5 & 4 & 0 \\ 3 & -2 & -1 & 3 & 1 \\ 0 & -3 & 1 & 2 & 0 \\ 1 & 2 & 0 & 2 & 4 \\ 6 & 1 & 2 & 1 & -1 \end{vmatrix}.$$

5. Find the adjoints of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix}$ and $\begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 1 & 4 & 3 \\ 2 & -2 & 0 & 1 \\ 5 & 1 & 6 & 2 \end{pmatrix}$

6. Find the inverses of the following matrices:

$$\begin{aligned} \text{(a)} & \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ -3 & 4 & -1 \end{pmatrix} \text{ over } \mathbb{Q}; & \text{(b)} & \begin{pmatrix} 0 & 1 & 3 \\ 5 & 4 & -2 \\ 2 & -1 & 1 \end{pmatrix} \text{ over } \mathbb{Q}; & \text{(c)} & \begin{pmatrix} \bar{1} & \bar{2} & \bar{0} \\ \bar{2} & \bar{1} & \bar{0} \\ \bar{0} & \bar{2} & \bar{2} \end{pmatrix} \text{ over } \mathbb{Z}_3; \\ \text{(d)} & \begin{pmatrix} \bar{1} & \bar{0} & \bar{7} & \bar{2} & \bar{3} \\ \bar{6} & \bar{9} & \bar{8} & \bar{1} & \bar{6} \\ \bar{8} & \bar{5} & \bar{9} & \bar{4} & \bar{10} \\ \bar{1} & \bar{0} & \bar{3} & \bar{2} & \bar{7} \\ \bar{2} & \bar{7} & \bar{1} & \bar{5} & \bar{4} \end{pmatrix} \text{ over } \mathbb{Z}_{11}; \end{aligned}$$

7. Let K be a field and $n \geq 2$. Prove that

$$\text{adjoint of (adjoint of } A) = (\det A)^{n-2}A$$

for any $A \in Mat_n(K)$.

8. Let K be a field and $n \geq 2$. Let $x, y \in K$ and put

$$d_n = \begin{vmatrix} x+y & xy & 0 & 0 & \dots \\ 0 & x+y & xy & 0 & \dots \\ 0 & 0 & x+y & xy & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

Express d_n in terms of d_{n-1} and d_{n-2} , and evaluate it in closed form.

9. Evaluate the determinant

$$\begin{vmatrix} \binom{c+m-1}{0} & \binom{c+m}{1} & \binom{c+m+1}{2} & \dots & \binom{c+m+m-1}{m} \\ \binom{c+m}{0} & \binom{c+m+1}{1} & \binom{c+m+2}{2} & \dots & \binom{c+m+m}{m} \\ \dots & \dots & \dots & \dots & \dots \\ \binom{c+m+m-1}{0} & \binom{c+m+m}{1} & \binom{c+m+m+1}{2} & \dots & \binom{c+m+m+m-1}{m} \end{vmatrix}.$$

10. Let $q \in \mathbb{N}$ and assume that K is a field of q elements. Using Theorem 44.21, find the orders of the groups $GL(n, K)$ and $SL(n, K)$ (cf. §17, Ex.17).