§45
Linear Equations

Let K be a field and $\alpha_{ij}, \beta_i \in K$ $(i = 1,2,\dots,m; j = 1,2,\dots,n)$. We ask if there are elements $x_1,x_2,\dots,x_n$ in K such that

$$(1) \qquad \begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= \beta_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= \beta_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n &= \beta_m. \end{aligned}$$

(1) is said to be a *system of linear equations*. We will not treat the general problem here. Our objective is this paragraph is to derive necessary and sufficient conditions for the solvability of (1) in the special case $m = n$. Concerning the case $m \neq n$, we will prove only the following consequence of Theorem 42.21.

45.1 Theorem: *Let K be a field and $\alpha_{ij} \in K$ $(i = 1,2,\dots,m; j = 1,2,\dots,n)$. If $n > m$, that is to say, if there are more unknowns than equations in the system*

$$(2) \qquad \begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= 0 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n &= 0, \end{aligned}$$

then there are elements $x_1,x_2,\dots,x_n$ in K, not all of them being zero, which satisfy the system (2).

Proof: Of course $x_1 = x_2 = \cdots = x_n = 0$ is a solution of (2), called the trivial solution. We ask whether nontrivial solutions of (2) exit. The claim is that there does exist nontrivial solutions of (2) when $n > m$.

Let $A = (\alpha_{ij}) \in Mat_n(K)$. Setting $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in Mat_{n\times 1}(K)$ and

$0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in Mat_{m\times 1}(K)$, we may write (2) as a matrix equation:

$$AX = 0.$$

The problem is thus: given $A \in Mat_{m\times n}(K)$, is there a nonzero X in $Mat_{n\times 1}(K)$ such that $AX = 0 \in Mat_{m\times 1}(K)$?

Since $A(N + M) = AN + AM$ and $A(\alpha N) = \alpha(AN)$ for any $N,M \in Mat_{n\times 1}(K)$ and $\alpha \in K$, the mapping

$$\begin{aligned} \varphi\colon Mat_{n\times 1}(K) &\to Mat_{m\times 1}(K) \\ N \quad &\to AN \end{aligned}$$

is a vector space homomorphism. From Theorem 42.21, we obtain

$$\begin{aligned} n = dim_K Mat_{n\times 1}(K) &= dim_K Ker\,\varphi + dim_K Im\,\varphi \\ &\leqslant dim_K Ker\,\varphi + dim_K Mat_{m\times 1}(K) \\ &= dim_K Ker\,\varphi + m, \end{aligned}$$

so

$$dim_K Ker\,\varphi \geqslant n - m > 0,$$
$$Ker\,\varphi \neq \{0\} \subseteq Mat_{n\times 1}(K),$$

and there does exist an $X \neq 0$ in $Ker\,\varphi$. So there is a nonzero $X \in Mat_{n\times 1}(K)$ with $AX = 0$, as was to be shown. □

45.2 Theorem: *Let K be a field,* $(\alpha_{ij}) \in Mat_n(K)$ *and let* $\beta_1, \beta_2, \ldots, \beta_n$ *be elements of K. If* $det\,(\alpha_{ij}) \neq 0$*, then the system*

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= \beta_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= \beta_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nn}x_n &= \beta_n \end{aligned} \tag{3}$$

has a unique solution in K, given by

$$x_j = \frac{det\,B_j}{det\,(\alpha_{ij})} \qquad (j = 1,2,\ldots,n),$$

where B_j *is the matrix obtained from* (α_{ij}) *by replacing its j-th column by*

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

Proof: Let $A = (\alpha_{ij})$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in Mat_{n\times 1}(K)$ and $B = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \in Mat_{n\times 1}(K)$. Then (3) can be written as a matrix equation:

$$AX = B. \tag{4}$$

Multiplying both sides of (4) on the left by $A^{-1} = \frac{1}{det\, A}$ (adjoint of $A)^{t}$, we obtain

$$X = \frac{1}{det\, A}\,(\text{adjoint of } A)^{t}B. \tag{5}$$

Also, multiplying both sides of (5) on the left by A, and using Theorem 44.12, we obtain (4). Thus (4) and (5) are equivalent. So the system (3) or (4) has a unique solution given by (5). In more detail, when we write A_{ij} for the cofactor of α_{ij} in A, so that (adjoint of A) = (A_{ij}), the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{det\, A} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$= \frac{1}{det\, A} \begin{pmatrix} A_{11}\beta_1 + A_{21}\beta_2 + \cdots + A_{n1}\beta_n \\ A_{12}\beta_1 + A_{22}\beta_2 + \cdots + A_{n2}\beta_n \\ \cdots\cdots\cdots \\ A_{1n}\beta_1 + A_{2n}\beta_2 + \cdots + A_{nn}\beta_n \end{pmatrix}.$$

So $x_j = \frac{1}{det\, A}(\beta_1 A_{1j} + \beta_2 A_{2j} + \cdots + \beta_n A_{nj})$ for $j = 1,2, \ldots ,n$. Comparing the expression in parentheses with the expansion

$$\alpha_{1j}A_{1j} + \alpha_{2j}A_{2j} + \cdots + \alpha_{nj}A_{nj}$$

(Theorem 44.15) of $det\, (\alpha_{ij})$ along the j-th column, we see that the paren-thetical expression is the expansion, along the j-th column, of the determinant of the matrix B_j that is obtained from (α_{ij}) by replacing its j-th column by B. Thus

$$x_j = \frac{\boldsymbol{det}\ B_j}{\boldsymbol{det}\ (\alpha_{ij})} \qquad (j = 1,2, \ldots ,n),$$

as claimed. □

The formula $x_j = \dfrac{\boldsymbol{det}\ B_j}{\boldsymbol{det}\ (\alpha_{ij})}$ is known as Cramer's rule after G. Cramer (1704-1752).

45.3 Theorem: *Let K be a field,* $(\alpha_{ij}) \in Mat_n(K)$. *The system*

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= 0 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots& \\ \alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nn}x_n &= 0 \end{aligned} \qquad (6)$$

has a nontrivial solution in K (*i.e., a solution distinct from the obvious one* $x_1 = x_2 = \ldots = x_n = 0$), *if and only if det* $(\alpha_{ij}) = 0$.

Proof: If *det* $(\alpha_{ij}) \neq 0$, then the system has a unique solution by Theorem 45.2, which must be $x_1 = x_2 = \cdots = x_n = 0$, as follows also from Cramer's rule, for the numerator determinants, having a column consisting of zeroes only, are all equal to 0. Thus, if the system has a nontrivial solution in K, then *det* (α_{ij}) must be zero.

Suppose conversely that *det* $(\alpha_{ij}) = 0$. Then the columns of (α_{ij}) are linearly dependent over K (Theorem 44.21): There are elements $\beta_1,\beta_2, \ldots ,\beta_n$ in K, not all of them being zero, such that

$$\beta_1\begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{n1} \end{pmatrix} + \beta_2\begin{pmatrix} \alpha_{12} \\ \alpha_{22} \\ \vdots \\ \alpha_{n2} \end{pmatrix} + \cdots + \beta_n\begin{pmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \vdots \\ \alpha_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus $x_1 = \beta_1, x_2 = \beta_2, \ldots, x_n = \beta_n$ is a nontrivial solution of (6). □

45.4 Remark: The theorems in this paragraph are chiefly of theoretical interest. Finding solutions of specific systems by the methods described in this paragraph would be very tedious.

Exercises

1. Find all solutions of the following systems of linear equations:

(a)
$$\begin{aligned} 3x + 4y - 5z &= -1 \\ 2x - 3y + z &= 3 \\ 2x + y + 6z &= 0; \end{aligned}$$

(b)
$$\begin{aligned} 4x + y - 5z - u &= 1 \\ 6x + 2y - 3z + 3u &= 8 \\ -4x + 5y - 2z + u &= -3 \\ 2x - 7z - 3u &= 0. \end{aligned}$$

2. Using Cramer's rule, find the solutions in $\mathbb{Z}_{13}$ of the following systems of linear equations, where $\overline{}$ denotes residue classes modulo 13:

(a)
$$\begin{aligned} \overline{2}x + \overline{11}y + \overline{4}z &= \overline{1} \\ \overline{3}x + \overline{8}y + \overline{5}z &= \overline{6} \\ \overline{9}x + \overline{12}y + \overline{4}z &= \overline{7}; \end{aligned}$$

(b)
$$\begin{aligned} \overline{2}x + \overline{11}y + \overline{4}z + \overline{3}u &= \overline{5} \\ \overline{8}x + \overline{10}y + \overline{6}z + \overline{7}u &= \overline{2} \\ \overline{1}x + \overline{9}y + \overline{2}z + \overline{8}u &= \overline{6} \\ \overline{3}x + \overline{1}y + \overline{0}z + \overline{5}u &= \overline{4}. \end{aligned}$$