Math 322 Final Examination

Name : _____ May 31, 1999

Signature : _____ 9:00–12:00

(1) Show that a polynomial $f(x) \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$ if and only if $f(x+1)$ is irreducible in $\mathbb{Z}[x]$. Using this, prove that for any odd prime number p, the pth cyclotomic polynomial $\Phi_p(x)$ is irreducible in $\mathbb{Z}[x]$. (15 points)

(2) Consider $\mathbb{R}$ and $\mathbb{C}$ as vector spaces over $\mathbb{R}$. Find all $\mathbb{R}$-vector space homomorphisms from $\mathbb{R}$ into $\mathbb{C}$. (10 points)

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(3) Let V be a vector space over a field K. Let W be a subgroup of the additive group $(V, +)$. Find necessary and sufficient conditions on W in order that the factor group V/W become a K-vector space under the multiplication by scalars $\alpha(v + W) := (\alpha v) + W$, where $\alpha \in K$, $v \in V$. (15 points)

(4) Let x, y be two distinct indeterminates over a field K. Let E be an extension of K, and let a, $b \in E$. If a, b are transcendental over K, prove or disprove that $K(a, b) \cong K(x, y)$. (10 points)

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(5) Find all prime numbers p such that there exist exactly 3 monic irreducible polynomials of degree 2 in $\mathbb{F}_p[x]$. (10 points)

(6) Is $f(x) := x^4 + 2x^3 + 4x^2 + 3x + 2 \in \mathbb{F}_5[x]$ irreducible in $\mathbb{F}_5[x]$. (15 points)

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(7) Describe the elements of the fields $\mathbb{F}_7[\sqrt{3}]$ and $\mathbb{F}_7[\sqrt{5}]$. Are these fields isomorphic? If so, find an isomorphism from $F_7[\sqrt{5}]$ onto $F_7[\sqrt{3}]$. (15 points)

(8) Consider the subfields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ of $\mathbb{R}$. Prove that, if $\{\alpha, \beta\}$ is a $\mathbb{Q}$-linearly independent subset of $\mathbb{Q}(\sqrt{2})$, then $\{\alpha, \beta\}$ is $\mathbb{Q}(\sqrt{3})$-linearly independent. (10 points)