In this paragraph, we examine an important subgroup of $S_n$, called the alternating group on $n$ letters. We begin with a definition that will play an important role throughout this paragraph.

**16.1 Definition:** A cycle of length 2 in $S_n$ (where $n \geq 2$) is called a **transposition**.

A transposition is therefore a permutation of the form $(ab)$ and has order 2 (Theorem 15.11). We remark that $(ab) = (ba)$.

**16.2 Theorem:** Any permutation in $S_n$ (where $n \geq 2$) can be written as a product of transpositions.

**Proof:** Since any permutation in $S_n$ can be written as a product of (disjoint) cycles (Theorem 15.9), it suffices to prove that any cycle can be written as a product of transpositions. This follows from $(abc \ldots e) = (ab)(ac)\ldots(ae)$ for cycles of length $> 1$. Also $1 = (12)(12)$ is a product of transpositions. This completes the proof. \hfill $\Box$

There is no uniqueness claim in Theorem 16.2. A permutation can be written as a product of different transpositions. For instance,


is written as a product of different transpositions. Nor is the number of transpositions is unique. The permutation $(132546)$ can be written as a product of five or nine transpositions:

In fact, we can attach a product of two transpositions \((ab)(ab) = \iota\) at will and increase the number of transpositions by 2. Hence a product of \(n\) transpositions can be written also as a product of \(n + 2, n + 4, n + 6, \ldots\) transpositions. We note that this does not change the parity of the number of transpositions. The parity of the number of transpositions is unique. If a permutation can be written as a product of an odd (even) number of transpositions, then, in any representation of this permutation as a product of transpositions, the number of transpositions is odd (even). A permutation cannot be written as a product of an odd number of transpositions and also as a product of an even number of transpositions. We proceed to prove this assertion. We need the notion of inversions of a permutation.

Let \(\sigma \in S_n\). We write \(\sigma\) in double row notation, where, in the first row, the numbers 1, 2, \ldots, \(n\) are in their natural order:

\[
\begin{pmatrix}
1 & 2 & \ldots & n \\
\sigma_1 & \sigma_2 & \ldots & \sigma_n \\
\end{pmatrix}.
\]

Corresponding to the correct inequalities

\[
1 < 2 \quad 1 < 3 \quad \ldots \quad 1 < n \\
2 < 3 \quad \ldots \quad 2 < n \\
\ldots \ldots \ldots \ldots \\
(n - 1) < n
\]

among the numbers in the first row, we obtain the inequalities

\[
\sigma_1 < \sigma_2 \quad \sigma_1 < \sigma_3 \quad \ldots \quad \sigma_1 < \sigma_n \\
\sigma_2 < \sigma_3 \quad \ldots \quad \sigma_2 < \sigma_n \\
\ldots \ldots \ldots \ldots \\
(n - 1)\sigma < n\sigma
\]

among the numbers in the second row when we replace each \(k\) by \(k\sigma\) \((k = 1, 2, \ldots, n)\). These inequalities will be referred to as the inequalities of \(\sigma\). In general, some of the inequalities of \(\sigma\) will be correct, some will be wrong (if \(\sigma \neq \iota\), there will be a wrong inequality of \(\sigma\)). A wrong inequality \(i\sigma < j\sigma\) of \(\sigma\) means: \(i < j\) but \(i\sigma > j\sigma\) i.e., the natural order of \(i\) and \(j\) is inverted in the second row (that is, the larger one precedes the smaller one). We call each wrong inequality of \(\sigma\) an inversion of \(\sigma\). For example, \(\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 3 & 1 & 4 \end{pmatrix}\) has the inequalities
eight of which are wrong, namely $2 < 1$, $5 < 3$, $5 < 1$, $5 < 4$, $6 < 3$, $6 < 1$, $6 < 4$, $3 < 1$. Hence there are eight inversions of $\sigma$.

The main work of this paragraph is done in the next lemma.

16.3 Lemma: Let $n \geq 2$, $\sigma \in S_n$ and let $(ik)$ be a transposition in $S_n$. If $\sigma$ has an odd number of inversions, then $(ik)\sigma$ has an even number of inversions. If $\sigma$ has an even number of inversions, then $(ik)\sigma$ has an odd number of inversions.

Proof: Since $(ik) = (ki)$, we assume, without loss of generality, that $i < k$. We have

$$\sigma = (1 \ldots i \ldots k \ldots n), \quad (ik)\sigma = (1 \ldots i \ldots k \ldots n).$$

The second rows of $\sigma$ and $(ik)\sigma$ are identical, aside from the locations of $i\sigma$ and $k\sigma$. Here $\sigma$ gives rise to the inequalities

1. $h\sigma < i\sigma$, $h\sigma < k\sigma$ where $h \in \{1, \ldots, i - 1\} =: H$,
2. $i\sigma < j\sigma$, $i\sigma < k\sigma$ where $j \in \{i + 1, \ldots, k - 1\} =: J$,
3. $i\sigma < m\sigma$, $i\sigma < k\sigma$ where $m \in \{k + 1, \ldots, n\} =: M$,

and to certain other inequalities that do not involve $i\sigma$ or $k\sigma$. And $(ik)\sigma$ gives rise to the inequalities

1. $h\sigma < k\sigma$, $h\sigma < i\sigma$ where $h \in H$,
2. $k\sigma < j\sigma$, $k\sigma < i\sigma$ where $j \in J$,
3. $k\sigma < m\sigma$, $j\sigma < i\sigma$ where $m \in M$,
4. $i\sigma < m\sigma$, $j\sigma < i\sigma$ where $m \in M$,
and to certain other inequalities that do not involve \(i\sigma\) or \(k\sigma\).

In the cases \(i = 1, k = i + 1, k = n\), there holds respectively \(H = \emptyset, J = \emptyset, M = \emptyset\) and the corresponding inequalities should be deleted. This does not impair the argument below.

We are to show that the number of inversions of \(\sigma\) and the number of inversions of \((ik)\sigma\) differ by an odd number.

The inequalities of \(\sigma\) and of \((ik)\sigma\) that do not involve \(i\sigma\) or \(k\sigma\) are identical. Also, the inequalities 1., 2., 3. of \(\sigma\) and \((ik)\sigma\) are the same (or absent). So only the inequalities

I. \(i\sigma < j\sigma, \quad i\sigma < k\sigma, \quad j\sigma < k\sigma\) (where \(j \in J\) of \(\sigma\))

and

II. \(k\sigma < j\sigma, \quad k\sigma < i\sigma, \quad j\sigma < i\sigma\) (where \(j \in J\) of \((ik)\sigma\))

are different. We must prove that the number of wrong inequalities in I and II differ by an odd number.

Since one of \(i\sigma < k\sigma, k\sigma < i\sigma\) is correct and the other is wrong, we must prove only that the number of wrong inequalities in

A. \(i\sigma < j\sigma, \quad j\sigma < k\sigma\) (where \(j \in J\))

and in

B. \(k\sigma < j\sigma, \quad j\sigma < i\sigma\) (where \(j \in J\))

differ by an even number.

Suppose there are \(s\) wrong inequalities \(i\sigma < j\sigma\) and \(t\) wrong inequalities \(j\sigma < k\sigma\) in A, where \(|J| \geq s \geq 0\) and \(|J| \geq t \geq 0\) (including the case \(J = \emptyset, |J| = 0\)). Then there are \(s + t\) wrong inequalities and there are

\(|J| - s) + (|J| - t) = 2|J| - (s + t)\) correct inequalities in A. Since B consists of the negations of the inequalities in A, there are \(2|J| - (s + t)\) wrong inequalities in B. So

\[
\text{(no of wrong inequalities in A)} - \text{(no of wrong inequalities in B)} = (s + t) - (2|J| - (s + t)) = 2(s + t - |J|) = \text{an even number.}
\]

This completes the proof. \(\square\)

**16.4 Definition:** Let \(n \in \mathbb{N}\) and let \(\sigma \in S_n\). If \(\sigma\) has an odd number of inversions, then \(\sigma\) is called an **odd permutation**. If \(\sigma\) has an even number of inversions, then \(\sigma\) is called an **even permutation**.
As the number of inversions of a permutation is uniquely determined, it is clear that a permutation cannot be both odd and even. With this terminology, Lemma 16.3 reads as follows.

**16.3 Lemma:** Let $n \geq 2$ and $\sigma \in S_n$. Let $(ik)$ be a transposition in $S_n$. If $\sigma$ is odd, then $(ik)\sigma$ is even. If $\sigma$ is even, then $(ik)\sigma$ is odd.

Applying Lemma 16.3 $r$ times, we have

**16.5 Lemma:** Let $n \geq 2$, $\sigma \in S_n$ and let $\tau_1, \tau_2, \ldots, \tau_r$ be transpositions in $S_n$. If $r$ is odd, then $\sigma$ and $\tau_1 \tau_2 \ldots \tau_r \sigma$ have the opposite "parity" (i.e., one of them is odd, the other is even). If $r$ is even, then $\sigma$ and $\tau_1 \tau_2 \ldots \tau_r \sigma$ have the same "parity".

**16.6 Theorem:** Let $n \geq 2$, $\pi \in S_n$. Then $\pi$ is an odd (even) permutation if and only if $\pi$ can be written as a product of an odd (even) number of transpositions. In particular, $\pi$ cannot be written as a product of an odd number of transpositions and also as a product of an even number of transpositions.

**Proof:** We use Lemma 16.5 with $\sigma = \iota$. Let $\pi$ be written as a product of transpositions, say $\pi = \tau_1 \tau_2 \ldots \tau_r$. Lemma 16.5 tells us that $\pi = \tau_1 \tau_2 \ldots \tau_r \iota$ and $\iota$ have opposite or same "parities" according as whether $r$ is odd or even. Since $\iota$ has 0 inversions, $\iota$ is an even permutation. So $\pi = \tau_1 \tau_2 \ldots \tau_r$ is an odd permutation or an even permutation according as whether $r$ is an odd number or an even number. The other assertion follows from the remark made after Definition 16.4.

We describe the "parity" of a product.
16.7 Theorem: Let $n \geq 2$. The product of two permutations in $S_n$ has the "parity" given by the following law.

\[
\begin{align*}
(\text{odd})(\text{odd}) &= (\text{even}) & (\text{odd})(\text{even}) &= (\text{odd}) \\
(\text{even})(\text{odd}) &= (\text{odd}) & (\text{even})(\text{even}) &= (\text{even}).
\end{align*}
\]

Proof: Let $\sigma, \pi \in S_n$. We want to find the "parity" of $\sigma \pi$. Let $\sigma = \tau_1 \tau_2 \cdots \tau_s$ and $\pi = \tau'_1 \tau'_2 \cdots \tau'_p$, where $\tau_1, \tau_2, \ldots, \tau_s, \tau'_1, \tau'_2, \ldots, \tau'_p$ are transpositions (Theorem 16.2). Then $\sigma \pi = \tau_1 \tau_2 \cdots \tau_s \tau'_1 \tau'_2 \cdots \tau'_p$ is a product of $s + p$ transpositions.

If $\sigma$ is an odd permutation and $\pi$ is an odd permutation, then $s$ is an odd number and $p$ is an odd number (Theorem 16.6), so $s + p$ is an even number, so $\sigma \pi$ is an even permutation (Theorem 16.6). Thus (odd)(odd) = (even). The other cases are proved similarly. □

The assertion of Theorem 16.7 resembles the rule for finding the sign of a product of two real numbers: the product of a negative number by a negative number is positive, etc. In order to exploit this analogy, we introduce a new term.

16.8 Definition: Let $n \in \mathbb{N}$ and $\sigma \in S_n$. The sign of $\sigma$ is the integer 1 or -1. We write $E(\sigma)$ for the sign of $\sigma$, and define it as follows.

\[
E(\sigma) = \begin{cases} 
1 & \text{if } \sigma \text{ is an even permutation} \\
-1 & \text{if } \sigma \text{ is an odd permutation.}
\end{cases}
\]

With this definition, the content of Theorem 16.7 can be expressed more succinctly.

16.7 Theorem: For any $\sigma, \pi$ in $S_n$, there holds $E(\sigma \pi) = E(\sigma)E(\pi)$.

□

16.9 Theorem: Let $n \geq 2$. The number of odd permutations in $S_n$ is equal to the number of even permutations in $S_n$. This number is $n!/2$. 

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Proof: We must find a one-to-one correspondence between the set of odd permutations and the set of even permutations in $S_n$. Now

$$T: \{ \sigma \in S_n: E(\sigma) = -1 \} \rightarrow \{ \sigma \in S_n: E(\sigma) = 1 \}$$

$$\sigma \rightarrow (12)\sigma$$

is a one-to-one mapping (by Lemma 8.1(1)) from the set of odd permutations in $S_n$ into the set of even permutations in $S_n$ (by Lemma 16.3), which is in fact onto, since any even permutation $\pi$ is the image, under $T$, of the odd permutation $(12)\pi$ (Lemma 16.3). So $T$ is a one-to-one correspondence between these sets and they contain equal number of elements, say $k$ elements. Since these sets are disjoint, and their union is $S_n$, there are $2k$ elements in $S_n$, whose order is $n!$ by Theorem 15.2. Hence $k = n!/2$.

\[\square\]

Theorem 16.7 asserts that the set of even permutations in $S_n$ is closed under multiplication. So it is a subgroup of $S_n$ by Lemma 9.3(2).

16.10 Definition: The subgroup of even permutations in $S_n$ ($n \geq 2$) is called the alternating group (on $n$ letters) and is written as $A_n$.

16.11 Theorem: For $n \geq 2$, $A_n$ is a group of order $n!/2$.

Proof: Theorem 16.9.

\[\square\]

Exercises

1. Find the sign of $(13524)$ and of $(153462)$.

2. Show that a cycle of length $m$ is odd (even) if and only if $m$ is even (odd).

3. Prove that $E(\sigma_1\sigma_2\ldots\sigma_t) = E(\sigma_1)E(\sigma_2)\ldots E(\sigma_t)$ for all permutations $\sigma_1, \sigma_2, \ldots, \sigma_t$ in $S_n$. 

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4. Find the sign of \((143)(1245)(243)\) and of \((1435)(25643)\) without evaluating these products.

5. Write all elements in \(A_2 \cdot A_3 \cdot A_4\).

6. Construct multiplication tables of \(A_2 \cdot A_3 \cdot A_4\).

7. Find all subgroups of \(A_4\). Does \(A_4\) have a subgroup of order 6?

8. Verify Lemma 16.3 by going through the argument in its proof in the specific cases below.

\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}, (ik) = (12), (14), (23), (26), (27), (67). \]