The reader might have missed the familiar quotient rule \((\frac{f}{g})' = \frac{f'g - fg'}{g^2}\) in Lemma 35.15. It was missing because \(\frac{f}{g}\) is not a polynomial.

We now introduce these quotients \(\frac{f}{g}\).

**36.1 Definition:** Let \(D\) be an integral domain and \(x, x_1, x_2, \ldots, x_n\) indeterminates over \(D\). Then \(D[x]\) and \(D[x_1,x_2, \ldots, x_n]\) are integral domains (Lemma 33.6, Lemma 33.10). An element in the the field of fractions of \(D[x]\) is called a *rational function (in \(x\)) over \(D\)*. The field of fractions of \(D[x]\) will be called the *field of rational functions over \(D\) (in \(x\))*, and will be denoted by \(D(x)\). An element in the the field of fractions of \(D[x_1,x_2, \ldots, x_n]\) is called a *rational function (in \(x_1,x_2, \ldots, x_n\)) over \(D\)*. The field of fractions of \(D[x_1,x_2, \ldots, x_n]\) will be called the *field of rational functions over \(D\) (in \(x_1,x_2, \ldots, x_n\))*, and will be denoted by \(D(x_1,x_2, \ldots, x_n)\).

Thus a rational function over \(D\) is a fraction \(\frac{f}{g}\) of two polynomials over \(D\), with \(g \neq 0\). Two rational functions \(\frac{f_1}{g_1}\) and \(\frac{f_2}{g_2}\) are equal if and only if the polynomials \(f_1g_2\) and \(g_1f_2\) are equal. Two rational functions \(\frac{f_1}{g_1}\) and \(\frac{f_2}{g_2}\) are added and multiplied according to the rules

\[
\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1g_2 + g_1f_2}{g_1g_2}, \quad \frac{f_1}{g_1} \cdot \frac{f_2}{g_2} = \frac{f_1f_2}{g_1g_2}.
\]

Here \(g_1\) and \(g_2\) are distinct from the zero polynomial over \(D\).

This terminology is unfortunate and misleading, because a rational function is *not* a function in the sense of Definition 3.1. A rational function is *not* a function of the 'rational' kind, whatever that might mean. The technical term we defined is *rational function*, a term consisting of two words "rational" and "function". The meaning of the
words "rational" and "function" do not play any role in Definition 36.1. A rational function is a fraction of polynomials over $D$. The reader should exercise caution about this point. One should not conclude that
\[
\frac{x^2 - 1}{x - 1} \quad \text{and} \quad \frac{x + 1}{1}
\]
in $\mathbb{C}(x)$ are different rational functions, on grounds that their domains are different, since the domain of the first one does not contain 1, whereas 1 is in the domain of the second one. Neither of them has a domain, for neither of them is a function. And these rational functions are equal because the polynomials $(x^2 - 1)1$ and $(x - 1)(x + 1)$ in $\mathbb{C}[x]$ are equal.

36.2 Lemma: Let $D$ be an integral domain and $F$ the field of fractions of $D$. Let $x$ be an indeterminate over $D$. Then $D(x) = F(x)$.

Proof: $F$ consists of the fractions $\frac{a}{b}$, where $a, b \in D$ and $b \neq 0$; and $D(x)$ consists of the fractions
\[
\frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0},
\]
where $a_n, a_{n-1}, \ldots, a_1, a_0, b_m, b_{m-1}, \ldots, b_1, b_0 \in D$ and the denominator is distinct from the zero polynomial in $D[x]$. Finally, $F(x)$ consists of the fractions
\[
\frac{c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0}{d_m x^m + d_{m-1} x^{m-1} + \cdots + d_1 x + d_0},
\]
where $c_n, c_{n-1}, \ldots, c_1, c_0, d_m, d_{m-1}, \ldots, d_1, d_0 \in F$ and the denominator is distinct from the zero polynomial in $F[x]$.

An element of $D$ is identified with the fraction $\frac{a}{1}$ in $F$ (Theorem 31.5), whence $D \subseteq F$. Thus $D[x] \subseteq F[x]$ as sets. Note that two elements $\frac{f(x)}{g(x)}$ and $\frac{p(x)}{q(x)}$ of $D(x)$ are equal in $D(x)$ if and only if $f(x)q(x) = g(x)p(x)$ in $D[x]$, and this holds if and only if $f(x)q(x) = g(x)p(x)$ in $F[x]$, so if and only if $\frac{f(x)}{g(x)}$
and \( \frac{p(x)}{q(x)} \) are equal in \( F(x) \). Thus every element of \( D(x) \) is in \( F(x) \) and equality in \( D(x) \) coincides with equality in \( F(x) \). So \( D(x) \subseteq F(x) \).

Next we show \( F(x) \subseteq D(x) \). Let \( \frac{p(x)}{q(x)} \in F(x) \), with \( p(x), q(x) \in F[x] \), \( q(x) \neq 0 \).

Then \( p(x) = \sum_{i=0}^{m} a_i x^i \), \( q(x) = \sum_{j=0}^{n} c_j x^j \), where \( a_i, b_i, c_j, d_j \in D \), \( b_i \neq 0 \), \( d_j \neq 0 \) for all \( i, j \) and not all of \( c_j \) are equal to \( 0 \in D \). We put \( b = b_0 b_1 \ldots b_{n-1} b_n \) and \( d = d_0 d_1 \ldots d_{m-1} d_m \). Then \( dbp(x) \) and \( dbq(x) \) are polynomials in \( D[x] \), and hence \( \frac{p(x)}{q(x)} = \frac{dbp(x)}{dbq(x)} \in D(x) \). So \( F(x) \subseteq D(x) \). This proves \( D(x) = F(x) \). □

As an illustration of Lemma 36.2, observe that

\[
\frac{2}{3} \frac{x^2 - \frac{1}{7} x + \frac{1}{4}}{x^2 + \frac{1}{3} x - \frac{1}{2}} \in \mathbb{Q}(x)
\]

is equal to the rational function

\[
\frac{5(56x^2 - 12x + 21)}{14(12x^2 + 10x - 15)} \in \mathbb{Z}[x].
\]

### 36.3 Remark:

Let \( D \) be an integral domain and \( F \) the field of fractions of \( D \). Then

\[
D(x_1, x_2, \ldots, x_n) = \text{field of fractions of } D[x_1, x_2, \ldots, x_n] = \text{field of fractions of } D[x_1, x_2, \ldots, x_{n-1}][x_n] = D[x_1, x_2, \ldots, x_{n-1}](x_n) = D(x_1, x_2, \ldots, x_{n-1})(x_n)
\]

by Lemma 36.2, with \( D[x_1, x_2, \ldots, x_{n-1}], D(x_1, x_2, \ldots, x_{n-1}), x_n \) in place of \( D,F,x \), respectively.

Also, we have \( D(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) \), for this is true when \( n = 1 \) (Lemma 36.2) and, when it is true for \( n = k \), so that \( D(x_1, x_2, \ldots, x_k) = F(x_1, x_2, \ldots, x_k) \), it is also true for \( n = k + 1 \):

\[
D(x_1, x_2, \ldots, x_k, x_{k+1}) = D(x_1, x_2, \ldots, x_k)(x_{k+1}) = F(x_1, x_2, \ldots, x_k)(x_{k+1}) = F(x_1, x_2, \ldots, x_k, x_{k+1}),
\]

the last equation by the remark above, with \( F \) in place of \( D \) and \( k + 1 \) in place of \( n \).

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In the remainder of this paragraph, we discuss partial fraction expansions of rational functions.

**36.4 Lemma:** Let $K$ be a field and let $f(x)$ be a nonzero polynomial in $K[x]$. Let $q(x)$, $r(x)$ be two nonzero, relatively prime polynomials of positive degree in $K[x]$. Suppose $\deg f(x) < \deg q(x)r(x)$ and suppose that $f(x)$ is relatively prime to $q(x)r(x)$. Then there are uniquely determined nonzero polynomials $a(x), b(x)$ in $K[x]$ such that

$$a(x)r(x) + b(x)q(x) = f(x), \quad \deg a(x) < \deg q(x), \quad \deg b(x) < \deg r(x).$$

**Proof:** We first prove the existence of $a(x)$ and $b(x)$. Since $q(x)$, $r(x)$ are relatively prime, there are polynomials $h(x)$, $k(x)$ in $K[x]$ with

$$h(x)r(x) + k(x)q(x) = 1.$$  

Multiplying both sides of this equation by $f(x)$ and putting $A(x) = f(x)h(x)$, $B(x) = f(x)k(x)$, we obtain

$$A(x)r(x) + B(x)q(x) = f(x).$$

We now divide $A(x)$ by $q(x)$ and $B(x)$ by $r(x)$:

$$A(x) = s(x)q(x) + a(x), \quad a(x) = 0 \text{ or } \deg a(x) < \deg q(x),$$

$$B(x) = u(x)r(x) + b(x), \quad b(x) = 0 \text{ or } \deg b(x) < \deg r(x).$$

Thus

$$a(x)r(x) + b(x)q(x) = (A(x) - s(x)q(x))r(x) + (B(x) - u(x)r(x))q(x)$$

$$= (A(x)r(x) + B(x)q(x)) - (s(x) + u(x))q(x)r(x)$$

$$= f(x) - (s(x) + u(x))q(x)r(x).$$

We claim $s(x) + u(x)$ is the zero polynomial in $K[x]$. Otherwise, we would have

$$\deg (s(x) + u(x)) \geq 0,$$

$$\deg (s(x) + u(x))q(x)r(x) \geq \deg q(x)r(x),$$

and since by hypothesis $\deg f(x) < \deg q(x)r(x)$,

$$\deg f(x) - (s(x) + u(x))q(x)r(x) \geq \deg q(x)r(x),$$

so that $a(x)r(x) + b(x)q(x) \neq 0$; in particular, both $a(x)$ and $b(x)$ cannot be zero. Assume, without loss of generality, that $a(x) \neq 0$ in case one of $a(x)$, $b(x)$ is zero and that $\deg a(x)r(x) \geq \deg b(x)q(x)$ in case neither of them is zero. Then we get the contradiction

$$\deg [f(x) - (s(x) + u(x))q(x)r(x)] = \deg (a(x)r(x) + b(x)q(x))$$

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\[ \deg a(x) + \deg r(x) \]
\[ = \deg a(x) + \deg r(x) \]
\[ < \deg q(x) + \deg r(x) \]
\[ = \deg q(x)r(x). \]

Thus \( s(x) + u(x) \), and consequently \( (s(x) + u(x))q(x)\) \( r(x) \) is the zero polynomial in \( K[x] \). This gives \( a(x)r(x) + b(x)q(x) = f(x) \). It remains to show that \( a(x) \) and \( b(x) \) are distinct from the zero polynomial in \( K[x] \). Both of them cannot be 0, for then \( f(x) \) would be also 0, which it is not by hypothesis. If one of them is 0, say if \( a(x) = 0 \), then \( b(x) \neq 0 \) and \( f(x) = b(x)q(x) \) would not be relatively prime to \( q(x)r(x) \) (because \( q(x) \) is of positive degree, so not a unit in \( K[x] \)), against the hypothesis. This proves the existence of \( a(x), b(x) \).

It remains to show the uniqueness of \( a(x) \) and \( b(x) \). If we have also

\[ a_1(x)r(x) + b_1(x)q(x) = f(x), \quad \deg a_1(x) < \deg q(x), \quad \deg b_1(x) < \deg r(x), \]

we obtain

\[
0 = f(x) - f(x) = (a_1(x)r(x) + b_1(x)q(x)) - (a(x)r(x) + b(x)q(x))
\]

\[
= (a(x) - a_1(x))r(x) - (b_1(x) - b(x))q(x),
\]

so

\[
(a(x) - a_1(x))r(x) = (b_1(x) - b(x))q(x). \quad (*)
\]

Hence

\[
r(x) \mid (b_1(x) - b(x))q(x) \quad \text{ in } K[x]
\]

\[
r(x) \mid b_1(x) - b(x) \quad \text{ in } K[x] \text{ as } r(x) \text{ and are } q(x) \text{ relatively prime.}
\]

Now \( b(x) \neq b_1(x) \) implies \( b(x) - b_1(x) \neq 0 \) and this gives

\[
\deg r(x) \leq \deg (b_1(x) - b(x)) \leq \max\{\deg b_1(x), \deg b(x)\} \leq \deg r(x),
\]

a contradiction. Thus \( b(x) = b_1(x) \) and we get then \( a(x) = a_1(x) \) from \((*)\).

So \( a(x) \) and \( b(x) \) are uniquely determined. \( \square \)

### 36.5 Lemma

Let \( K \) be a field and let \( \frac{f(x)}{g(x)} \) be a nonzero rational function in \( K(x) \), with \( \deg f(x) < \deg g(x) \). Suppose that \( f(x) \) and \( g(x) \) are both monic and that \( f(x) \) is relatively prime to \( g(x) \). Assume \( g(x) = q(x)r(x) \), where \( q(x)r(x) \) are two relatively prime polynomials of positive degree in \( K[x] \). Then there are uniquely determined nonzero polynomials \( a(x), b(x) \) in \( K[x] \) such that

\[
\frac{f(x)}{g(x)} = \frac{f(x)}{q(x)r(x)} = \frac{a(x)}{q(x)} + \frac{b(x)}{r(x)}
\]

and

\[
\deg a(x) < \deg q(x), \quad \deg b(x) < \deg r(x).
\]
Proof: If \( \frac{f(x)}{g(x)} \) is a nonzero rational function in \( K(x) \), then \( f(x) \) is a nonzero polynomial in \( K[x] \), and \( f(x) \) is relatively prime to \( g(x) = q(x)r(x) \). As \( f(x) \) and \( g(x) \) are monic, these conditions determine \( f(x) \) and \( g(x) \) uniquely. The polynomials \( q(x) \), \( r(x) \) are relatively prime and \( \deg f(x) \) is smaller than \( \deg q(x)r(x) \). So the hypotheses of Lemma 36.4 are satisfied and therefore there are uniquely determined nonzero polynomials \( a(x), b(x) \) in \( K[x] \) such that

\[
f(x) = a(x)r(x) + b(x)q(x),
\]

and

\[
\deg a(x) < \deg q(x), \quad \deg b(x) < \deg r(x).
\]

Dividing both sides of the equation above by \( g(x) = q(x)r(x) \), we see that there are uniquely determined nonzero polynomials \( a(x), b(x) \) in \( K[x] \) such that

\[
\frac{f(x)}{g(x)} = \frac{f(x)}{q(x)r(x)} = \frac{a(x)}{q(x)} + \frac{b(x)}{r(x)}
\]

and

\[
\deg a(x) < \deg q(x), \quad \deg b(x) < \deg r(x).
\]

By induction on \( m \), we obtain the following lemma.

36.6 Lemma: Let \( K \) be a field and let \( \frac{f(x)}{g(x)} \) be a nonzero rational function in \( K(x) \), with \( \deg f(x) < \deg g(x) \). Suppose that \( f(x) \) and \( g(x) \) are both monic and that \( f(x) \) is relatively prime to \( g(x) \). Assume \( g(x) = q_1(x)q_2(x)\ldots q_m(x) \), where \( q_1(x), q_2(x), \ldots , q_m(x) \) are pairwise relatively prime monic polynomials of positive degree in \( K[x] \). Then there are uniquely determined nonzero polynomials \( a_1(x), a_2(x), \ldots , a_m(x) \) in \( K[x] \) such that

\[
f(x) = \frac{f(x)}{q_1(x)q_2(x)\ldots q_m(x)} = \frac{a_1(x)}{q_1(x)} + \frac{a_2(x)}{q_2(x)} + \cdots + \frac{a_m(x)}{q_m(x)}
\]

and

\[
\deg a_i(x) < \deg q_i(x) \quad \text{for all } i = 1, 2, \ldots , m.
\]

36.7 Lemma: Let \( K \) be a field and \( x \) an indeterminate over \( K \). Let \( g(x) \) be a polynomial in \( K[x] \) of degree \( \geq 1 \). Then, for any \( f(x) \in K[x] \), there are uniquely determined polynomials \( r_0(x), r_1(x), r_2(x), \ldots , r_n(x) \) such that
\[ f(x) = r_0(x) + r_1(x)g(x) + r_2(x)g(x)^2 + \cdots + r_n(x)g(x)^n \]

and
\[ r_i(x) = 0 \quad \text{or} \quad \deg r_i(x) < \deg g(x) \quad \text{for all} \quad i = 1, 2, \ldots, n. \]

**Proof:** From \( \deg g \geq 1 \), we know that \( g \neq 0 \). So we may divide \( f \) by \( g \) and obtain \( f = q_0g + r_0 \), where \( q_0, r_0 \in K[x] \), with \( r_0 = 0 \) or \( \deg r_0 < \deg g \). Here \( q_0 \) and \( r_0 \) are uniquely determined by \( f \) and \( g \) (Theorem 34.4) and we have \( f = r_0 + q_0g \). If \( q_0 = 0 \), we are done (with \( n = 0 \)). Otherwise, since \( f = q_0g + r_0 \), \( \deg g \geq 1 \) and \( r_0 = 0 \) or \( \deg r_0 < \deg g \), we have \( \deg q_0 < \deg f \) (Lemma 33.3). We now divide \( q_0 \) by \( g \) and obtain \( q_0 = q_1g + r_1 \), where \( q_1, r_1 \in K[x] \), with \( r_1 = 0 \) or \( \deg r_1 < \deg g \). Here \( q_1 \) and \( r_1 \) are uniquely determined by \( q_0 \) and \( g \) (hence by \( f \) and \( g \)) and \( f = r_0 + r_1g + q_1g^2 \). If \( q_1 = 0 \), we are done. Otherwise, \( \deg q_1 < \deg q_0 \). We then divide \( q_1 \) by \( g \) and obtain \( q_1 = q_2g + r_2 \), where \( q_2, r_2 \in K[x] \), with \( r_2 = 0 \) or \( \deg r_2 < \deg g \). Here \( q_2 \) and \( r_2 \) are uniquely determined by \( q_1 \) and \( g \) (hence by \( f \) and \( g \)) and \( f = r_0 + r_1g + r_2g^2 + q_2g^3 \). If \( q_2 = 0 \), we are done. Otherwise, we have \( \deg q_2 < \deg q_1 \). We continue this process. As the degrees of \( q_0, q_1, q_2, \ldots \) get smaller and smaller, this process cannot go on indefinitely. Sooner or later, we will meet a \( q_n \) equal to 0 \( \in K[x] \). Then, with uniquely determined \( r_0, r_1, r_2, \ldots, r_n \), we have \( f = r_0 + r_1g + r_2g^2 + \cdots + r_ng^n \), where \( r_i(x) = 0 \) or \( \deg r_i < \deg g \) for all \( i = 1, 2, \ldots, n \).

\[ \square \]

In the situation of Lemma 36.7, the unique expression
\[ f = r_0 + r_1g + r_2g^2 + \cdots + r_ng^n \]
of \( f(x) \), where \( r_i(x) = 0 \) or \( \deg r_i < \deg g \) for all \( i = 1, 2, \ldots, n \), is called the \( g \)-adic expansion of \( f \).

**36.8 Theorem:** Let \( K \) be a field and \( \frac{p(x)}{q(x)} \) a nonzero rational function in \( K(x) \), where \( p(x), q(x) \in K[x] \) are relatively prime in \( K[x] \). Let \( u \) be the leading coefficient of \( q(x) \) and let \( q(x) = u g_1(x)^{m_1} g_2(x)^{m_2} \cdots g_t(x)^{m_t} \) be the decomposition of \( q(x) \) into polynomials irreducible over \( K \), where \( g_i(x) \) are monic. Then there are uniquely determined polynomials \( G(x) \),
\( a_1^{(1)}(x), a_2^{(1)}(x), \ldots, a_{m_1}^{(1)}(x), a_1^{(2)}(x), a_2^{(2)}(x), \ldots, a_{m_2}^{(2)}(x), \ldots, a_1^{(t)}(x), a_2^{(t)}(x), \ldots, a_{m_t}^{(t)}(x) \) in \( K[x] \) such that

\[
\frac{p(x)}{q(x)} = G(x) + \frac{a_1^{(1)}(x)}{g_1(x)} + \frac{a_2^{(1)}(x)}{g_1^2(x)} + \cdots + \frac{a_{m_1}^{(1)}(x)}{g_1^{m_1}(x)} + \frac{a_1^{(2)}(x)}{g_2(x)} + \frac{a_2^{(2)}(x)}{g_2^2(x)} + \cdots + \frac{a_{m_2}^{(2)}(x)}{g_2^{m_2}(x)} + \cdots + \frac{a_1^{(t)}(x)}{g_t(x)} + \frac{a_2^{(t)}(x)}{g_t^2(x)} + \cdots + \frac{a_{m_t}^{(t)}(x)}{g_t^{m_t}(x)}
\]

and \( \deg a_i^{(k)}(x) \leq \deg g_k(x) \) or \( a_i^{(k)}(x) = 0 \) for all \( i \) and \( k \).

**Proof:** We divide \( p(x) \) by \( q(x) \) and find unique polynomials \( G(x), H(x) \) in \( K[x] \) with \( p(x) = q(x)G(x) + H(x) \), \( \deg H(x) < \deg q(x) \) or \( H(x) = 0 \). In the latter case, everything is proved (\( a_i^{(k)}(x) = 0 \) for all \( i \) and \( k \)). If \( H(x) \neq 0 \), let \( \nu \) be the leading coefficient of \( H(x) \) and put \( c = \nu/u \). Then \( H(x) \) and \( q(x) \) are relatively prime (since \( p(x) \) and \( q(x) \) are). We have \( H(x) = \nu h(x) \), where \( h(x) \) is monic, relatively prime to \( q(x) \) and

\[
\frac{p(x)}{q(x)} = G(x) + \frac{h(x)}{q(x)}
\]

with \( \deg h(x) < \deg q(x) \). We may use Lemma 36.6 and get uniquely determined nonzero polynomials \( b_1(x), b_2(x), \ldots, b_t(x) \) in \( K[x] \) such that

\[
\frac{h(x)}{q(x)} = \frac{b_1(x)}{g_1(x)^{m_1}} + \frac{b_2(x)}{g_2(x)^{m_2}} + \cdots + \frac{b_t(x)}{g_t(x)^{m_t}}
\]

and \( \deg b_k(x) < \deg g_k(x)^{m_k} \) for all \( k = 1, 2, \ldots, t \). We put \( f_k(x) = cb_k(x) \). Then

\[
\frac{p(x)}{q(x)} = G(x) + \frac{f_1(x)}{g_1(x)^{m_1}} + \frac{f_2(x)}{g_2(x)^{m_2}} + \cdots + \frac{f_t(x)}{g_t(x)^{m_t}}
\]

and, since \( c \) is uniquely determined by \( p(x) \) and \( q(x) \), the polynomials \( f_k(x) \) are also uniquely determined. Since

\[
\deg f_k(x) = \deg b_k(x) < \deg g_k(x)^{m_k},
\]

in the \( g_k(x) \)-adic expansion

\[
f_k(x) = r_0(x) + r_1(x)g_k(x) + r_2(x)g_k(x)^2 + \cdots + r_n(x)g_k(x)^n
\]

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of $f_k(x)$, the polynomials $r_s(x) = 0$ for $s \geqslant m_k$. So let

$$f_k(x) = a_1^{(k)}(x)g_k(x)^{m_k-1} + a_2^{(k)}(x)g_k(x)^{m_k-2} + \cdots + a_{m_k-1}^{(k)}(x)g_k(x) + a_m^{(k)}(x)$$

be the $g_k(x)$-adic expansion of $f_k(x)$. The polynomials $a_1^{(k)}, a_2^{(k)}, \ldots, a_{m_k}^{(k)}$ in $K[x]$ are uniquely determined and $\deg a_i^{(k)} < \deg g_k(x)$ or $a_i^{(k)} = 0$ for all $i = 1, 2, \ldots, m_k$. Hence, for all $k = 1, 2, \ldots, t$, there holds

$$\frac{f_k(x)}{g_k(x)^{m_k}} = \frac{a_1^{(k)}(x)}{g_k^{1}(x)} + \frac{a_2^{(k)}(x)}{g_k^{2}(x)} + \cdots + \frac{a_{m_k}^{(k)}(x)}{g_k^{m_k}(x)}$$

and this completes the proof.

The equation

$$\frac{p(x)}{q(x)} = G(x) + \frac{a_1^{(1)}(x)}{g_1^{1}(x)} + \frac{a_2^{(1)}(x)}{g_1^{2}(x)} + \cdots + \frac{a_{m_1}^{(1)}(x)}{g_1^{m_1}(x)}$$

$$+ \frac{a_1^{(2)}(x)}{g_2^{1}(x)} + \frac{a_2^{(2)}(x)}{g_2^{2}(x)} + \cdots + \frac{a_{m_2}^{(2)}(x)}{g_2^{m_2}(x)}$$

$$+ \cdots$$

$$+ \frac{a_1^{(r)}(x)}{g_t^{1}(x)} + \frac{a_2^{(r)}(x)}{g_t^{2}(x)} + \cdots + \frac{a_{m_t}^{(r)}(x)}{g_t^{m_t}(x)}$$

in Theorem 36.8 is known as the expansion of $\frac{p(x)}{q(x)}$ in partial fractions.

**Exercises**

1. Let $K$ be a field. For any nonzero rational function $\frac{f}{g}$ in $K(x)$, we define the *degree of $\frac{f}{g}$*, denoted by $\deg \frac{f}{g}$, by $\deg \frac{f}{g} = \deg f - \deg g$. Prove that the degree of a rational function is well defined. Can you extend the degree assertions in Lemma 33.3 to rational functions?

2. Let $K$ be a field. For any rational function $\frac{f}{g}$ in $K(x)$, we define the *derivative of $\frac{f}{g}$*, denoted by $\left(\frac{f}{g}\right)'$, by declaring
\[
\left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2}.
\]

Prove that differentiation is well defined, i.e., prove that \( \frac{f}{g} = \frac{a}{b} \) implies \( \left( \frac{f}{g} \right)' = \left( \frac{a}{b} \right)' \).

3. Extend Lemma 35.15 and Lemma 35.16 to derivatives of rational functions in one indeterminate over a field.

4. Expand \( \frac{2x^3 + 3x^2 + 8x + 6}{(x^3 + 3x + 3)(x^2 + 2x + 3)} \in \mathbb{Q}(x) \) and \( \frac{4x^3 + 3x^2 + x + 2}{x^5 + 4x^4 + 4x^3 + 2x + 2} \in \mathbb{Z}_5(x) \) in partial fractions.

5. Let \( K \) be a field and let \( a_1, a_2, \ldots, a_m \) be pairwise distinct elements in \( K \). Put \( g(x) = (x - a_1)(x - a_2) \ldots (x - a_m) \) and let \( f(x) \) be a nonzero polynomial in \( K[x] \) with \( \deg f(x) < m \). Show that

\[
\frac{f(x)}{g(x)} = \sum_{i=1}^{m} \frac{f(a_i)}{g'(a_i)} \frac{1}{x-a_i}.
\]